EXISTENCE AND UNIQUENESS THEOREMS FOR SOLUTIONS OF SOME TWO POINT BOUNDARY VALUE PROBLEMS FOR $y'' = f(x,y,y')$

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CHAPTER I

INTRODUCTION

In 1922 O. Perron [23] introduced a new technique in the investigation of the existence of solutions of the boundary value problem for Laplace's equation. This technique used the existence of solutions to boundary value problems for small circles and the existence for larger regions of subharmonic and superharmonic functions which satisfied certain differential inequalities.

Later in 1937 E. F. Beckenbach [7] generalized the concept of convex functions to the idea of functions subordinate to families of continuous real valued functions that have members passing through arbitrary finite points (x_1, y_1) and (x_2, y_2). These subordinate functions were called subfunctions or superfunctions if their graphs lay below or above respectively the graphs of certain members of these families. In 1949 M. M. Peixoto [22] applied Beckenbach's generalization to the study of functions subordinate to the family of solutions of the second order differential equation $y'' = f(x, y, y')$ with arbitrary finite boundary conditions. This is an extension of the idea that convex functions are subordinate to the family of solutions of the differential equation $y'' = 0$ with arbitrary boundary conditions. Peixoto showed that in order for a function $s(x)$ to be a subfunction with respect to $y'' = f(x,y,y')$, it must satisfy the differential inequality $s'' \geq f(x,s,s')$.

L. Fountain and L. Jackson [10] continued this investigation in 1962 by using this subordinate function approach and determined existence and uniqueness conditions for the boundary value problem

$$y'' = f(x,y,y')$$
$$y(a) = \alpha,\ y(b) = \beta$$

on compact intervals $[a,b]$ of the real line where α, β are finite real numbers. In analogy to Perron's technique, they employed the existence of solutions for sufficiently "close" boundary values of

$$y'' = f(x,y,y')$$
$$y(x_1) = y_1,\ y(x_2) = y_2$$

along with a maximum principle suggested from Beckenbach's work ([7], Theorem 4). They found conditions on $f(x,y,y')$ such that a "generalized" solution to $y'' = f(x,y,y')$ could be constructed and made this generalized solution satisfy the boundary conditions by a further restriction on $f(x,y,y')$. Soon afterwards, a student of Jackson's, J. W. Bebernes [2], replaced one of their major assumptions on the nature of

$f(x,y,y')$ by a purely analytical condition. He also weakened one of Fountain and Jackson's analytical requirements on $f(x,y,z)$.

The purpose of this study is to present a development of the subfunction approach for establishing existence and uniqueness of solutions to some two point boundary value problems for $y'' = f(x,y,y')$. Most of this development originated in the papers of Fountain and Jackson and Bebernes. Many of their results were stated without proof or with only an outline of proof. To make this study more self-contained, detailed proofs are provided for all results. For clarity, the symbol $\square$ is used to denote the end of a proof.

The establishment of the properties of subfunctions, superfunctions, and the generalized solution is accomplished by successively placing analytical requirements on $f(x,y,z)$. Throughout this development $f(x,y,z)$ is assumed to be defined, real valued, and continuous on the region

$$R_3 = \{(x,y,z) \mid a \leq x \leq b,\ |y| < \infty, |z| < \infty\}.$$

In several instances results are obtained under less restrictive assumptions for the special boundary value problem

$$y'' = g(x,y)$$
$$y(a) = \alpha,\ y(b) = \beta$$

where $g(x,y)$ is real valued and continuous on the region

$$R_2 = \{(x,y) \mid a \leq x \leq b, \ |y| < \infty\}.$$

To begin the development, the continuity conditions alone are used to obtain various existence results for two point boundary value problems of the type

$$y'' = f(x,y,y')$$
$$y(x_1) = y_1, \ y(x_2) = y_2$$

when $|y_1|$, $|y_2|$, and $|(y_2-y_1)/(x_2-x_1)|$ are bounded and $|x_2-x_1| > 0$ is sufficiently small. This is followed by the establishment of two forms of a maximum principle which require additional restrictions on $f(x,y,z)$.

These results are used to determine some properties of subfunctions and superfunctions which are necessary in establishing the concept of the generalized solution. Several existence conditions for subfunctions and superfunctions are provided by further restricting $f(x,y,z)$. Using these results, the generalized solution is defined and its properties are determined.

Finally, the generalized solution is shown to be a solution of $y'' = f(x,y,y')$ and to satisfy the boundary conditions when an additional restriction is placed on $f(x,y,z)$. The main result of this work will be the following theorem.

<u>Theorem</u> 21. Suppose that

(1) $f(x,y,y')$ is a nondecreasing function of y for each fixed x and y' in R_3 and satisfies a Lipschitz condition with respect to y' on each fixed compact subset of R_3,

(2) there exists a $K > 0$ for which

$$|f(x,0,y') - f(x,0,0)| \leq K\,|y'|$$

for $a \leq x \leq b$ and $|y'| < \infty$, and

(3) there is a positive continuous function $\upsilon(t)$ defined for $t \geq 0$ such that

$$|f(x,y,y') - f(x,y,0)| \leq K_T\,\upsilon\,(|y'|)$$

and

$$\int_0^\infty \frac{t}{\upsilon(t)+1}\,dt = +\infty$$

where K_T is a constant depending on compact subsets T of R_2 for $(x,y) \in T$ and $|y'| < \infty$.

Then the generalized solution is the solution of class $C^2[a,b]$ of the boundary value problem

$$y'' = f(x,y,y')$$
$$y(a) = \alpha,\ y(b) = \beta.$$

The present state of development of the subfunction and superfunction concepts for second order ordinary differential equations is not significantly advanced beyond Bebernes's work published in 1963. However, several ideas from this work have been useful both in pure form and in a related form, the concepts of lower and upper solutions of $y'' = f(x,y,y')$.

Working with the differential equation $y'' = g(x,y)$, Bebernes and Jackson [6] established existence and uniqueness conditions for solutions on infinite intervals using subfunctions and superfunctions. K. W. Schrader [25] at this same time employed the subfunction concept along with the existence of solutions of the boundary value problem $y'' = f(x,y,y')$, $y(a) = y(b) = 0$ (see Theorem 22), in studying the existence of solutions of $y'' = f(x,y,y')$ on $(-\infty,\infty)$.

Soon afterwards Bebernes and R. Gaines [4], [5] found existence and uniqueness conditions for the boundary value problem

$$\begin{gathered} y'' = f(x,y,y') \\ a_o y(a) + a_1 y'(a) = \alpha \\ b_o y(b) + b_1 y'(b) = \beta \end{gathered}$$

using the subfunction concept. They also showed that the unique solution $y(x;\alpha,\beta)$ of this boundary value problem depends continuously on the boundary data α and β.

In 1968 Jackson [14] published a summary work on the subfunction and superfunction and related concepts which contained most of the results in this area up to that time. In this paper he proved some relationships between the existence of solutions to initial value problems and boundary value problems for $y'' = f(x,y,y')$ on compact intervals assuming various uniqueness conditions.

Working along a different line of development, R. Mathsen [18], [19], [20] used subfunctions and superfunctions to prove some disconjugacy results for third order ordinary differential equations. In this work he was able to use results for subfunctions and superfunctions with respect to second order differential equations such as for the equation $(y')'' = F(x,0,(y'),(y')')$.

The main conceptual development since Bebernes's paper in 1963 has been the generalization of the subfunction to the third order differential equation $y''' = h(x,y,y',y'')$ on compact intervals $[a,b]$ by Jackson and Schrader [16]. The boundary conditions considered are of two types: $y(x_o) = y_o$, $y(x_1) = y_1$, $y(x_2) = y_2$ and either $y(x_o) = y_o$, $y'(x_o) = y_1$, $y(x_2) = y_2$ or $y(x_o) = y_o$, $y(x_2) = y_1$, $y'(x_2) = y_2$ where $[x_o,x_2] \subset [a,b]$. Because of the different types of boundary conditions, there are two kinds of subfunctions which can be shown to be of the same type under certain conditions. In this work the main assumptions are the continuity of $h(x,y,z,w)$, the uniqueness of solutions of three point boundary value problems on $[a,b]$, and the extension of solutions of initial value problems to $[a,b]$.

Other work has used the related concepts of lower and upper solutions which only need to satisfy certain second order differential inequalities and require no restrictions on $f(x,y,z)$. H. W. Knobloch [17], Schrader [15], [26], Jackson [15], [13], Bebernes and Fraker [3], and K. Schmitt [24] have found results such as the existence of sequences of solutions of $y'' = f(x,y,y')$ converging to solutions in the norm $|y| + |y'|$, existence of solutions on compact and infinite intervals, disconjugacy conditions, and existence of solutions on $[0,1]$ where $(0,y(0), y'(0))$ and $(1,y(1),y'(1))$ are contained in certain compact or closed connected sets. A frequently occurring assumption in this work with lower and upper solutions is an integral conditions similar to that used in Theorem 21.

Thus the subfunction and superfunction and their related concepts continue to be useful aids in the investigation of non-linear ordinary differential equations.

CHAPTER II

ASSUMPTIONS AND BASIC PRINCIPLES

The primary results needed to investigate the existence of solutions for the boundary value problem

$$y'' = f(x,y,y')$$
$$y(a) = \alpha,\ y(b) = \beta$$

using the subordinate function approach are established in this section. In addition to the continuity of $f(x,y,z)$ on R_3, the following assumptions may also be required of $f(x,y,z)$ to prove certain assertions.

A_1: $f(x,y,z)$ is a nondecreasing function of y for each fixed x and z in R_3.

A_2: $f(x,y,z)$ satisfies a Lipschitz condition with respect to z on each fixed compact subset of R_3.

Note that $g(x,y)$ is a nondecreasing function of y for each fixed x in R_2 is also included in assumption A_1.

The first of these fundamental results is an "existence in the small" lemma basically due to E. Picard. This plays the role of Perron's solutions to boundary value problems on small circles.

Lemma 1. Given any $M>0$ and $N>0$, there is a $\delta(M,N)>0$ such that the boundary value problem

$$\begin{aligned} y'' &= f(x,y,y') \\ y(x_1) &= y_1,\ y(x_2) = y_2 \end{aligned} \tag{2-1}$$

has a solution of class $C^{(2)}$ on $[x_1,x_2]$ for any points (x_1,y_1) and (x_2,y_2) with x_1,x_2 in the compact interval $[a,b]$, $|x_2-x_1| \leq \delta(M,N)$, $|y_1| \leq M$, $|y_2| \leq M$, and $|(y_2-y_1)/(x_2-x_1)| \leq N$.

Proof. Let $z(x)$ be a solution of

$$\begin{aligned} z'' &= f(x,z,z') \\ z(x_1) &= 0,\ z(x_2) = 0 \end{aligned}$$

and consider the differential equation

$$y''(x) = f(x,z(x)+px+q,z'(x)+p)$$

where p and q are real constants. Then setting $y(x)=z(x)+px+q$, suppose that $z(x)+px+q$ is a solution of the differential equation $y'' = f(x,y,y')$. In order to obtain the boundary values for (2-1), set $y_1 = z(x_1)+px_1+q = px_1+q$ and $y_2 = z(x_2)+px_2+q = px_2+q$. These conditions make $p = (y_2-y_1)/(x_2-x_1)$ and $q = y_1-px_1$.

Assume $x_1 \leq x_2$. Integration of $z''(x) = f(x,z(x)+px+q,\ z'(x)+p)$ results in

$$z'(x) = z'(x_1) + \int_{x_1}^{x} f(t,z(t)+pt+q,z'(t)+p)\,dt,$$

where $x \varepsilon [x_1, x_2]$. A second integration yields

$$z(x) = z'(x_1)(x-x_1) + \int_{x_1}^{x}\int_{x_1}^{t} f(s,z(s)+ps+q,z'(s)+p)dsdt$$

since $z(x_1) = 0$. Letting $x = x_2$ and using the boundary condition $z(x_2) = 0$,

$$z'(x_1) = -\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\int_{x_1}^{t} f(s,z(s)+ps+q,z'(s)+p)dsdt.$$

This makes

$$z(x) = -\frac{x-x_1}{x_2-x_1}\int_{x_1}^{x_2}\int_{x_1}^{t} f(s,z(s)+ps+q,z'(s)+p)dsdt$$

$$+ \int_{x_1}^{x}\int_{x_1}^{t} f(s,z(s)+ps+q,z'(s)+p)dsdt.$$

By reversing the order of integration for $x_1 \leq x \leq x_2$

$$\int_{x_1}^{x}\int_{x_1}^{t} f(s,z(s)+ps+q,z'(s)+p)dsdt$$

$$= \int_{x_1}^{x}\int_{s}^{x} f(s,z(s)+ps+q,z'(s)+p)dtds$$

$$= \int_{x_1}^{x} (x-s)f(s,z(s)+ps+q,z'(s)+p)ds$$

$$= \int_{x_1}^{x} (x-s)\ F(s)\ ds,$$

where $F(s) = f(s,z(s)+ps+q,z'(s)+p)$. Now the integral equation can be rewritten in the more compact form

$$z(x) = -\frac{x-x_1}{x_2-x_1}\int_{x_1}^{x_2}(x_2-s)F(s)ds + \int_{x_1}^{x}(x-s)F(s)ds$$

$$= -\frac{x-x_1}{x_2-x_1}\int_{x}^{x_2}(x_2-s)F(s)ds - \frac{x-x_1}{x_2-x_1}\int_{x_1}^{x}(x_2-s)F(s)ds$$

$$+ \int_{x_1}^{x} (x-s)F(s)ds$$

$$= \int_{x}^{x_2} \frac{(x_1-x)(x_2-s)}{x_2-x_1} F(s)ds + \int_{x_1}^{x} [- \frac{(x-x_1)(x_2-s)}{x_2-x_1}$$

$$+ x-s] F(s)ds$$

$$= \int_{x}^{x_2} \frac{(x_1-x)(x_2-s)}{x_2-x_1} F(s)ds + \int_{x_1}^{x} \frac{(x_2-x)(x_1-s)}{x_2-x_1} F(s)ds$$

$$= \int_{x_1}^{x_2} G(x,s)F(s)ds$$

$$= \int_{x_1}^{x_2} G(x,s)f(s,z(s)+ps+q,z'(s)+p)ds,$$

using the Green's function kernel $G(x,s)$ defined by

$$G(x,s) = \begin{cases} \dfrac{(x_2-x)(x_1-s)}{x_2-x_1} & \text{if } x_1 \le s \le x \le x_2 \\[2ex] \dfrac{(x_2-s)(x_1-x)}{x_2-x_1} & \text{if } x_1 \le x \le s \le x_2. \end{cases}$$

For any $x_1 \le s \le x \le x_2$

$$|G(x,s)| = \frac{(x_2-x)(s-x_1)}{x_2-x_1} \le \frac{(x_2-x)(x-x_1)}{x_2-x_1},$$

and similarly for any $x_1 \le x \le s \le x_2$

$$|G(x,s)| \le \frac{(x_2-x)(x-x_1)}{x_2-x_1}.$$

Let $\gamma(x) \equiv (x_2-x)(x-x_1)/(x_2-x_1)$ and then $|G(x,s)| \leq \gamma(x)$ for all $x_1 \leq x, s \leq x_2$. Since $\gamma'(x) = (-2x+x_1+x_2)/(x_2-x_1)$ and $\gamma''(x) = -2/(x_2-x_1) < 0$, then $\gamma(x)$ has a maximum value at $x = (x_1+x_2)/2$ so that

$$|G(x,s)| \leq \gamma\left(\frac{x_1+x_2}{2}\right) = \frac{x_2-x_1}{4}.$$

Define now the set of functions

$$\mathcal{F} = \{w(x) \mid w(x) \varepsilon C[x_1,x_2] \cap C^{(1)}[x_1,x_2],\ w(x_1)=w(x_2)=0, \text{ and } ||w|| \leq 2(M+N)\}$$

where

$$||w|| = \max\{\sup_{[x_1,x_2]} |w(x)|, \sup_{[x_1,x_2]} |w'(x)|\}$$

This set is nonempty because the horizontal line $\ell(x) = 0$ on $[x_1,x_2]$ is an element of $\mathcal{F}$. For $w \varepsilon \mathcal{F}$ and $x_1 \leq x \leq x_2$, let

$$T[w(x)] = \int_{x_1}^{x_2} G(x,s)f(s,w(s)+ps+q,w'(s)+p)ds.$$

This defines a mapping from $\mathcal{F}$ into $C[x_1,x_2]$. By the definition of $G(x,s)$, $G(x_1,s) = G(x_2,s) = 0$ which makes $T[w(x_1)] = T[w(x_2)] = 0$. Furthermore, using the substitution $F_1(s) = f(s,w(s)+ps+q,w'(s)+p)$,

$$D_xT[w(x)] = D_x\{\int_{x_1}^{x} \frac{(x_2-x)(x_1-s)}{x_2-x_1}F_1(s)ds + \int_{x}^{x_2} \frac{(x_2-s)(x_1-x)}{x_2-x_1}F_1(s)ds$$

$$= -\int_{x_1}^{x} \frac{x_1-s}{x_2-x_1}F_1(s)ds + \frac{(x_2-x)(x_1-x)}{x_2-x_1}F_1(x)$$

$$- \int_{x}^{x_2} \frac{x_2-s}{x_2-x_1}F_1(s)ds - \frac{(x_2-x)(x_1-x)}{x_2-x_1}F_1(x)$$

$$= \int_{x_1}^{x} \frac{s-x_1}{x_2-x_1}F_1(s)ds + \int_{x}^{x_2} \frac{s-x_2}{x_2-x_1}F_1(s)ds,$$

which is a continuous function on $[x_1,x_2]$. Note also that

$$D_x^{\,2}T[w(x)] = F_1(x) = f(x,w(x)+px+q,w'(x)+p) \qquad (2\text{-}2)$$

is continuous on $[x_1,x_2]$. Let L be the compact subset of R_3 specified by

$$L = \{(x,y,z) \mid x_1\le x\le x_2,\ px+q-2(M+N)(b-a)\le y\le px+q+2(M+N)(b-a),$$
$$|z|\le (2M+3N)/[1-2N(M+N)]\}$$

and let ρ be the maximum value of the continuous function $|f(x,y,z)|$ on L. Then for $x_1 \le x \le x_2$

$$|T[w(x)]| = |\int_{x_1}^{x_2}G(x,s)f(s,w(s)+ps+q,w'(s)+p)ds|$$

$$\le \int_{x_1}^{x_2}|G(x,s)||f(s,w(s)+ps+q,w'(s)+p)|ds$$

$$\le \int_{x_1}^{x_2} \frac{x_2-x_1}{4}\rho\ ds$$

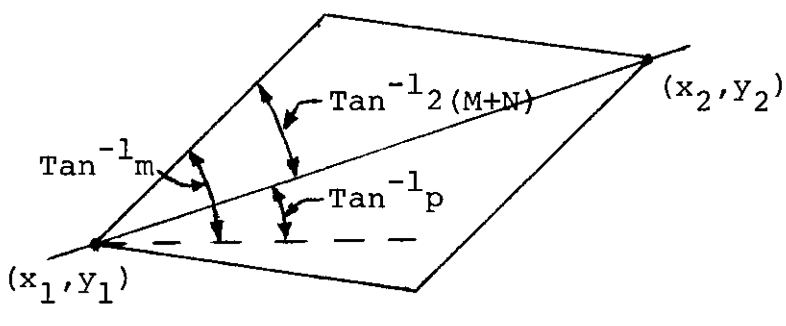
making

$$|T[w(x)]| \le \frac{\rho}{4}(x_2-x_1)^2 \tag{2-3}$$

and similarly

$$\begin{aligned}
|D_x T[w(x)]| &= \left|\int_{x_1}^{x} \frac{s-x_1}{x_2-x_1} f(s,w(s)+ps+q,w'(s)+p)\,ds \right.\\
&\qquad \left. + \int_{x}^{x_2} \frac{s-x_2}{x_2-x_1} f(s,w(s)+ps+q,w'(s)+p)\,ds \right|\\
&\le \int_{x_1}^{x} \left|\frac{s-x_1}{x_2-x_1}\right| |f(s,w(s)+ps+q,w'(s)+p)|\,ds\\
&\qquad + \int_{x_1}^{x_2} \left|\frac{s-x_2}{x_2-x_1}\right| |f(s,w(s)+ps+q,w'(s)+p)|\,ds\\
&\le \int_{x_1}^{x} \rho\,ds + \int_{x}^{x_2} \rho\,ds\\
&= \rho(x_2-x_1),
\end{aligned} \tag{2-4}$$

since the requirement that $w \epsilon F$ confines $w(x) + px + q$ to the rhombic shaped region shown below



where the angle addition formula for tangents makes

$$m = \frac{2(M+N)+p}{1-2(M+N)p} \leq \frac{2(M+N)+N}{1-2(M+N)N} .$$

This last inequality is true because slope m thought of as a function of slope p is an increasing function since

$$m'(p) = \frac{[1-2(M+N)p] - [-2(M+N)][2(M+N)+p]}{[1-2(M+N)p]^2} = \frac{1+4(M+N)^2}{[1-2(M+N)p]^2} \geq 0$$

and $p \leq N$. Hence, if $x_2 - x_1$ is smaller than $\delta(M,N) \equiv \min\{[8(M+N)/\rho]^{1/2}, 2(M+N)/\rho\}$, then

$$\begin{aligned} ||T[w(x)]|| &= \max\{\sup_{[x_1,x_2]} |T[w(x)]|, \sup_{[x_1,x_2]} |D_x T[w(x)]|\} \\ &\leq 2(M+N) \end{aligned}$$

which shows that $T:\mathcal{F}\to\mathcal{F}$.

Suppose $\{w_n\}_{n=1}^{\infty}$ is a sequence of functions in $\mathcal{F}$ that converge uniformly on $[x_1,x_2]$ to a continuous function $w(x)\varepsilon\mathcal{F}$. Let $\varepsilon>0$. By the continuity of $f(x,y,z)$, there is a $\sigma>0$ such that $|f(s,w(s)+ps+q,w'(s)+p) - f(s,w_n(s)+ps+q,w_n'(s)+p)| < \varepsilon$ whenever $||w-w_n|| < \sigma$. Because of the uniform convergence of $\{w_n\}$ to w, there is an $N_\sigma>0$ such that $||w-w_n|| < \sigma$ for all $n>N_\sigma$. Then if $n>N_\sigma$ and $x_1 \leq x \leq x_2$,

$$\begin{aligned} |T[w(x)] - T[w_n(x)]| = |\int_{x_1}^{x_2} G(x,s)[f(s,w(s)+ps+q,w'(s)+p) \\ -f(s,w_n(s)+ps+q,w_n'(s)+p)]ds| \end{aligned}$$

$$\leq \int_{x_1}^{x_2} |G(x,s)| \varepsilon \, ds$$

$$\leq \varepsilon \int_{x_1}^{x_2} \frac{x_2-x_1}{4} \, ds$$

$$= \varepsilon \frac{(x_2-x_1)^2}{4} \qquad (2\text{-}5)$$

and

$$|D_x T[w(x)] - D_x T[w_n(x)]| = \left|\int_{x_1}^{x} \frac{s-x_1}{x_2-x_1}[f(s,w(s)+ps+q,w'(s)+p)\right.$$

$$- f(s,w_n(s)+ps+q,w_n'(s)+p)]ds$$

$$+ \left|\int_{x}^{x_2} \frac{s-x_2}{x_2-x_1}[f(s,w(s)+ps+q,w'(s)+p)\right.$$

$$\left.-f(s,w_n(s)+ps+q,w_n'(s)+p)]ds\right|$$

$$\leq \int_{x_1}^{x} \left|\frac{s-x_1}{x_2-x_1}\right| \varepsilon \, ds + \int_{x}^{x_2} \left|\frac{s-x_2}{x_2-x_1}\right| \varepsilon \, ds$$

$$\leq \int_{x_1}^{x} \varepsilon \, ds + \int_{x}^{x_2} \varepsilon \, ds$$

$$= \varepsilon(x_2-x_1).$$

These two computations show that $||T[w(x) - T[w_n(x)]|| \leq \varepsilon \max\{(x_2-x_1)^2/4, x_2-x_1\}$. Thus $\{Tw_n\}_{n=1}^{\infty}$ converges uniformly to Tw on the compact interval $[x_1,x_2]$ and T is a continuous mapping on F.

Let $x_1 \leq t \leq \tau \leq x_2$. Then for any $w \in F$

$$|T[w(t)]-T[w(\tau)]| = \left|\int_{x_1}^{x_2} [G(t,s)-G(\tau,s)]f(s,w(s)+ps+q,w'(s)+p)ds\right|$$

$$\leq \int_{x_1}^{x_2} |G(t,s)-G(\tau,s)|\rho\ ds$$

$$= \frac{\rho}{x_2-x_1}\{\int_{x_1}^{t} |(x_2-t)(x_1-s) - (x_2-\tau)(x_1-s)|ds$$

$$+ \int_{t}^{\tau} |(x_2-s)(x_1-t) - (x_2-\tau)(x_1-s)|ds$$

$$+ \int_{\tau}^{x_2} |(x_2-s)(x_1-t) - (x_2-s)(x_1-\tau)|ds\}$$

$$= \frac{\rho}{x_2-x_1}\{\int_{x_1}^{t} |(x_1-s)(\tau-t)|ds + \int_{t}^{\tau} |(x_2-s)(x_1-t)$$

$$- (x_2-\tau)(x_1-s)|ds + \int_{\tau}^{x_2} |(x_2-s)(\tau-t)|ds\}$$

$$\leq \frac{\rho}{x_2-x_1}\{\int_{x_1}^{t} (s-x_1)(\tau-t)ds + \int_{t}^{\tau} [(x_2-s)(t-x_1)$$

$$+ (x_2-\tau)(s-x_1)]ds + \int_{\tau}^{x_2} (x_2-s)(\tau-t)ds\}$$

$$= \frac{\rho}{2(x_2-x_1)}\{(\tau-t)(t-x_1)^2 + (x_2-t)^2(t-x_1)$$

$$- (x_2-\tau)^2(t-x_1) + (x_2-\tau)(\tau-x_1)^2$$

$$- (x_2-\tau)(t-x_1)^2 + (x_2-\tau)^2(\tau-t)\}$$

$$= \frac{\rho}{2(x_2-x_1)}\{(\tau-t)(t-x_1)^2+(x_2-\tau)^2(\tau-t)+(t-x_1)[(x_2-t)^2$$

$$- (x_2-\tau)^2] + (x_2-\tau)[(\tau-x_1)^2 - (t-x_1)^2]\}$$

$$= \frac{\rho}{2(x_2-x_1)}\{(\tau-t)(t-x_1)^2 + (x_2-\tau)^2(\tau-t)$$

$$+ (t-x_1)[(x_2-t)+(x_2-\tau)][(x_2-t)-(x_2-\tau)]$$

$$+ (x_2-\tau)[(\tau-x_1)+(t-x_1)][(\tau-x_1)-(t-x_1)]\}$$

$$= \frac{\rho}{2(x_2-x_1)}\{(\tau-t)[(t-x_1)^2 + (x_2-\tau)^2]$$

$$+ (t-x_1)(2x_2-t-\tau)(\tau-t)$$

$$+ (x_2-\tau)(\tau+t-2x_1)(\tau-t)\}$$

$$\leq \frac{\rho}{2(x_2-x_1)}(\tau-t)\{2(x_2-x_1)^2 + (x_2-x_1)(2x_2-2x_1)$$

$$+ (x_2-x_1)(2x_2-2x_1)\}$$

$$= 3\rho(x_2-x_1)(\tau-t),$$

which shows that $\{Tw \mid w\varepsilon F\}$ is an equicontinuous set. From (2-3) this image set is bounded at every point of $[x_1,x_2]$. Therefore by the Schauder-Tychonoff Theorem ([9], page 9), the mapping T has a fixed point z(x) in F and $y(x) = z(x) + px + q$ is the solution of the boundary value problem (2-1) provided $|x_2-x_1| \leq \delta(M,N)$. By (2-2) and because z(x) is an element of F,

$$z''(x) = D_x^2 T[z(x)] = f(x,z(x)+px+q,z'(x)+p)$$

is continuous on $[x_1,x_2]$. Thus y(x) is the solution of class $C^{(2)}[x_1,x_2]$. □

From (2-3) and (2-4) for $x_1 \leq x \leq x_2$,

$$|y(x) - (px+q)| = |z(x)| = |T[z(x)]| \leq \frac{\rho}{4}(x_2-x_1)^2$$

and

$$|y'(x) - p| = |z'(x)| = |D_x T[z(x)]| \leq \rho(x_2-x_1).$$

Let $\varepsilon>0$ and then $|y(x)-(px+q)|\leq\varepsilon$ and $|y'(x)-p|\leq\varepsilon$ whenever $|x_2-x_1| \leq \min\{\delta(M,N), (4\varepsilon/\rho)^{1/2}, \varepsilon/\rho\}$. This gives the following corollary to the proof of Lemma 1.

<u>Lemma</u> 2. Let $M>0$, $N>0$ be fixed and let $\delta(M,N)$ be as in Lemma 1. Then given any $\varepsilon>0$ there is an η, $0<\eta\leq\delta(M,N)$, such that for any points (x_1,y_1) and (x_2,y_2) with x_1,x_2 in $[a,b]$, $|x_2-x_1|\leq\eta$, $|y_1|\leq M$, $|y_2|\leq M$, and $|(y_2-y_1)/(x_2-x_1)|\leq N$, there is a solution $y(x)$ of (2-1) of class $C^{(2)}$ on $[x_1,x_2]$ with $|y(x) - \omega(x)| \leq \varepsilon$ and $|y'(x) - \omega'(x)| \leq \varepsilon$ where $\omega(x)$ is the linear function with $\omega(x_1) = y_1$ and $\omega(x_2) = y_2$.

For the boundary value problem

$$\begin{gathered} y'' = g(x,y) \\ y(x_1) = y_1, \quad y(x_2) = y_2, \end{gathered} \tag{2-6}$$

a slightly less restrictive result holds true.

<u>Lemma</u> 3. Given any $M>0$ there is a $\delta(M)>0$ such that the boundary value problem (2-6) has a solution $y(x)$ of class $C^{(2)}$ on $[x_1,x_2]$ for any points (x_1,y_1) and (x_2,y_2) with x_1,x_2 in $[a,b]$, $|x_2-x_1|\leq \delta(M)$, $|y_1|\leq M$, and $|y_2|\leq M$. Further, given any $\varepsilon>0$ there is an η, $0<\eta\leq\delta(M)$, such that if $|x_2-x_1|\leq\eta$, then $|y(x) - \omega(x)| \leq \varepsilon$ and $|y'(x) - \omega'(x)| \leq \varepsilon$.

<u>Proof</u>. This is proved in exactly the same way as for Lemma 1 except for the following simplifications. Using the new notation

$$F^* = \{w(x) \mid w(x) \varepsilon C[x_1,x_2] \cap C^{(1)}[x_1,x_2],\ w(x_1)=w(x_2)=0,$$
$$\text{and } ||w||^* \leq 2M\},$$

$$||w||^* = \sup_{[x_1,x_2]} |w(x)|,$$

$$L^* = \{\ (x,y) \mid x_1 \leq x \leq x_2,\ px+q-2M \leq y \leq px+q+2M\},$$

$$\rho^* = \sup_{(x,y)\varepsilon L^*} |g(x,y)|,$$

gives that for $x_1 \leq x \leq x_2$ and $w \varepsilon F^*$

$$|T[w(x)]| = \left| \int_{x_1}^{x_2} G(x,s)g(s,w(s)+ps+q)ds \right| \leq \frac{\rho^*}{4}(x_2-x_1)^2. \qquad (2\text{-}3^*)$$

Then if $|x_2-x_1| \leq \delta(M) \equiv (8M/\rho^*)^{1/2}$, this makes $T:F^* \to F^*$. Furthermore,

$$|D_x T[w(x)]| \leq \rho^*(x_2-x_1). \qquad (2\text{-}4^*)$$

A second simplification to the proof of Lemma 1 is that (2-5) is sufficient to show that T is a continuous mapping on F^*. $\square$

Now to begin another line of development, let $\theta(x)$ be any function defined on a compact interval [c,d]. For any $x_0 \varepsilon (c,d)$ define

$$\overline{D}\theta(x_0) = \limsup_{\delta\to 0} \frac{\theta(x_0+\delta) - \theta(x_0-\delta)}{2\delta}$$

and

$$\underline{D}\theta(x_0) = \liminf_{\delta\to 0} \frac{\theta(x_0+\delta) - \theta(x_0-\delta)}{2\delta}$$

where the limit superior and limit inferior are defined in the usual sense ([1], pages 353-354). Using these differential quantities, several relationships between functions satisfying certain differential inequalities can be exhibited for compact intervals $[c,d] \subset [a,b]$. The first is the following maximum principle requiring a strict inequality.

<u>Lemma</u> 4. If $f(x,y,y')$ satisfies condition A_1 and if the functions $\phi(x)$ and $\psi(x)$ satisfy:

(i) ϕ, ψ are of class $C[c,d] \cap C^{(1)}(c,d)$,

(ii) on (c,d), $\underline{D}\phi'(x) \geq f(x,\phi(x),\phi'(x))$ and $\overline{D}\psi'(x) \leq f(x,\psi(x),\psi'(x))$ with at least one of these being strict inequality on (c,d),

(iii) for $M \geq 0$, $\phi(c) - \psi(c) \leq M$ and $\phi(d) - \psi(d) \leq M$,

then $\phi(x) - \psi(x) < M$ for all $x \varepsilon (c,d)$.

<u>Proof</u>. Consider first the case $M=0$. Suppose for some points in (c,d) that $\phi(x) - \psi(x) \geq 0$. Because $[c,d]$ is compact and $\phi(x) - \psi(x)$ is continuous on $[c,d]$, there is a point $x_0 \varepsilon (c,d)$ at which $\phi(x) - \psi(x)$ takes on its maximum value. This means that $\phi'(x_0) = \psi'(x_0)$. Then

$$\begin{aligned}
\underline{D}[\phi'(x_0) - \psi'(x_0)] &\geq \underline{D}\phi'(x_0) - \overline{D}\psi'(x_0) \\
&> f(x_0,\phi(x_0),\phi'(x_0)) - f(x_0,\psi(x_0),\psi'(x_0)) \\
&= f(x_0,\phi(x_0),\phi'(x_0)) - f(x_0,\psi(x_0),\phi'(x_0)) \\
&\geq 0
\end{aligned}$$

by condition A_1 and since $\phi(x_0) \geq \psi(x_0)$. However, because $\underline{D}[\phi'(x_0) - \psi'(x_0)] > 0$, the First Derivative Test for extrema of functions implies that $\phi(x_0) - \psi(x_0)$ is a minimum value for $\phi(x) - \psi(x)$ which is a contradiction. Thus $\phi(x) - \psi(x) < 0$ on (c,d).

If $M>0$, set $\psi_1(x) = \psi(x) + M$. Then the assumption that $\phi(c) - \psi(c) \leq M$ makes $\phi(c) - \psi_1(c) = \phi(c) - [\psi(c) + M] \leq 0$ and $\phi(d) - \psi_1(d) \leq 0$. For any $x\varepsilon(c,d)$

$$\begin{aligned}\overline{D}\psi_1'(x) &= \overline{D}\psi'(x) \\ &\leq f(x,\psi(x),\psi'(x)) \\ &\leq f(x,\psi(x)+M,\psi'(x)) \\ &= f(x,\psi_1(x), \psi_1'(x)).\end{aligned}$$

Then the preceding case gives that $\phi(x) - \psi_1(x) < 0$ or $\phi(x) - \psi(x) < M$ on (c,d). □

The next two results are useful in eliminating the necessity of the strict inequality.

<u>Lemma</u> 5. If $f(x,y,z)$ satisfies A_1 and A_2 and if there is a function ϕ such that

(i) $\phi\varepsilon C^{(1)}[c,d]$,

(ii) on (c,d), $\underline{D}\phi'(x) \geq f(x,\phi(x),\phi'(x))$,

then given $\varepsilon>0$ there is a function $\phi_1\varepsilon C^{(1)}[c,d]$ such that

$$\phi(x) - \varepsilon \leq \phi_1(x) \leq \phi(x)$$

on $[c,d]$ and $\underline{D}\phi_1'(x) > f(x,\phi_1(x),\phi_1'(x))$ on (c,d).

<u>Proof</u>. Since $\phi(x)$ and $\phi'(x)$ are continuous on the compact interval $[c,d]$, there is an $M>0$ for which $|\phi(x)| \leq M$ and

$|\phi'(x)| \leq M$ for all $c \leq x \leq d$. Let $R_3^* = \{(x,y,z) | c \leq x \leq d, |y| \leq M+1, |z| \leq M+1\}$. By condition A_2 there is a $K^* > 0$ such that

$$|f(x,y,y_1') - f(x,y,y_2')| \leq K^* |y_1 - y_2|$$

for all points $(x,y,y_1'),(x,y,y_2')$ in R_3^*. Now let $z(x)$ be a solution of

$$z'' - (K^* + 1)z' = 0$$

on $[c,d]$. Then for constant c_1, $z'(x) = c_1 e^{(K^*+1)x}$ and choosing $c_1 = -\varepsilon e^{-(K^*+1)d}$ makes $-\varepsilon \leq z'(x) < 0$ on $[c,d]$. Integration of $z'(x)$ results in

$$z(x) = \frac{c_1}{K^*+1} e^{(K^*+1)x} + c_2 = \frac{-\varepsilon}{K^*+1} e^{(K^*+1)(x-d)} + c_2$$

for integration constant c_2. Set $c_2 = \varepsilon$ which makes

$$z(x) = \varepsilon[1 - \frac{1}{K^*+1} e^{(K^*+1)(x-d)}].$$

If $c \leq x \leq d$, then $0 < (K^*+1)^{-1} e^{(K^*+1)(x-d)} < 1$ which means that $0 \leq z(x) \leq \varepsilon$. Using this solution

$$\begin{aligned}
&\underline{D}[\phi'(x)-z'(x)]-f(x,\phi(x)-z(x),\phi'(x)-z'(x)) \\
&\quad \geq f(x,\phi(x),\phi'(x))-f(x,\phi(x)-z(x),\phi'(x)-z'(x))-z''(x) \\
&\quad \geq f(x,\phi(x)-z(x),\phi'(x))-f(x,\phi(x)-z(x),\phi'(x)-z'(x))-z''(x) \\
&\quad \geq -K^*|z'(x)|-(K^*+1)z'(x)
\end{aligned}$$

because for $\varepsilon \leq 1$ this means $|\phi(x)-z(x)| \leq |\phi(x)|+|z(x)| \leq M+1$

and $|\phi'(x)-z'(x)| \leq M+1$. Finally,

$$\underline{D}[\phi'(x)-z'(x)] - f(x,\phi(x)-z(x),\phi'(x)-z'(x))$$
$$\geq K^*z'(x) - (K^*+1)z'(x)$$
$$= - z'(x)$$
$$> 0$$

or $\underline{D}[\phi'(x)-z'(x)] > f(x,\phi(x)-z(x),\phi'(x)-z'(x))$. Thus $\phi_1(x) = \phi(x) - z(x)$ is a function having the desired properties. $\square$

Lemma 5'. If $f(x,y,y')$ satisfies A_1 and A_2 and if there is a function ψ such that

(i) $\psi \varepsilon C^{(1)}[c,d]$,

(ii) on (c,d), $\overline{D}\psi'(x) \leq f(x,\psi(x),\psi'(x))$,

then given $\varepsilon>0$ there is a function $\psi_1 \varepsilon C^{(1)}[c,d]$ such that

$$\psi(x) \leq \psi_1(x) \leq \psi(x) + \varepsilon$$

on $[c,d]$ and $\overline{D}\psi_1'(x) < f(x,\psi_1(x),\psi_1'(x))$ on (c,d).

Proof. This is done in an analogous manner to the method used in proving Lemma 5 using $\psi_1(x) = \psi(x) + z(x)$ for $z(x) = \varepsilon[1 - (K^*+1)^{-1}e^{-(K^*+1)(x-c)}]$. $\square$

These previous three lemmas are now used to establish the following very important maximum principle.

Lemma 6. If $f(x,y,y')$ satisfies A_1 and A_2 and if there exist functions ϕ and ψ satisfying:

(i) ϕ,ψ are in $C[c,d] \cap C^{(1)}(c,d)$,

(ii) on (c,d), $\underline{D}\phi'(x) \geq f(x,\phi(x),\phi'(x))$ and

$\overline{D}\psi'(x) \leq f(x,\psi(x),\psi'(x))$,

(iii) for some $M\geq 0$, $\phi(c) - \psi(c) \leq M$ and $\phi(d) - \psi(d) \leq M$, then $\phi(x) - \psi(x) \leq M$ on $[c,d]$.

Proof. First consider the case $M=0$. Suppose that $\phi(x) - \psi(x) > 0$ for some points in (c,d). By the continuity of $\phi(x)$ and $\psi(x)$ on $[c,d]$, there is an $x_0 \varepsilon (c,d)$ at which $\phi(x_0) - \psi(x_0) = N = \max\limits_{[c,d]}\{\phi(x) - \psi(x)\} > 0$. Further there exist $c<x_1<x_2<d$ such that $x_0 \varepsilon (x_1,x_2)$ and $\phi(x_i) - \psi(x_i) \leq N/2$ for $i=1,2$. Choose $0 < \varepsilon \leq N/2$. Because $\psi \varepsilon C^{(1)}[x_1,x_2]$, then by Lemma 5' there is a $\psi_1 \varepsilon C^{(1)}[x_1,x_2]$ with $\psi(x) \leq \psi_1(x) \leq \psi(x)+\varepsilon$ and $\overline{D}\psi_1(x) < f(x,\psi_1(x),\psi_1'(x))$ on (x_1,x_2). Now for $i=1,2$

$$\phi(x_i) - \psi_1(x_i) \leq \phi(x_i) - \psi(x_i) \leq \frac{N}{2}$$

which means by Lemma 4 that $\phi(x) - \psi_1(x) < N/2$ on (x_1,x_2). In particular,

$$\phi(x_0) < \psi_1(x_0) + \frac{N}{2} \leq \psi(x_0) + \varepsilon + \frac{N}{2} \leq \psi(x_0) + N$$

Contradicting the assumed role of x_0. Thus $\phi(x) - \psi(x) \leq 0$ on $[c,d]$.

For the case $M>0$, let $\psi_2(x)=\psi(x)+M$. Using the same reasoning as in the second part of the proof of Lemma 4, $\phi(x) - \psi_2(x) \leq 0$ on $[c.d]$. Therefore $\phi(x) - \psi(x) \leq M$ on $[c,d]$. $\square$

CHAPTER III

SUBFUNCTIONS AND SUPERFUNCTIONS

This section is devoted to introducing and developing the concept of the subfunction and its dual form, the superfunction. Subfunctions with respect to the solutions of the differential equation

$$y'' = f(x,y,y') \tag{3-1}$$

are a generalization of convex functions in R_2. All the ideas developed here are restricted to compact subintervals $[c,d]$ of $[a,b]$.

Definition 1. A real valued function $s(x)$ defined on $[c,d]$ is said to be a subfunction with respect to (3-1) in case $s(x) \leq y(x)$ on $[x_1,x_2]$ for any $[x_1,x_2] \subset [c,d]$ and any solution $y(x)$ of (3-1) on $[x_1,x_2]$ with $s(x_1) \leq y(x_1)$ and $s(x_2) \leq y(x_2)$.

Definition 2. A real valued function $S(x)$ defined on $[c,d]$ is said to be a superfunction with respect to (3-1) in case $S(x) \geq y(x)$ on $[x_1,x_2]$ whenever $S(x_1) \geq y(x_1)$ and $S(x_2) \geq y(x_2)$.

The following development is concerned primarily with the case of the subfunction. Where there is a dual result for the superfunction, it is stated without proof and labeled in an analogous manner.

<u>Theorem</u> 1. If $s(x)$ is a subfunction with respect to (3-1) on $[c,d]$, then the right-hand and left-hand limits, $s(x_0+0)$ and $s(x_0-0)$, exist at every x_0 in (c,d) and the appropriate one-sided limits exist at $x = c$ and $x = d$. These limits may be infinite.

<u>Proof</u>. Let x_0 be an element of (c,d) and suppose that $s(x_0+0)$ does not exist. Then there exist finite numbers p and q such that $\liminf_{x\to x_0+} s(x) \leq p < q \leq \limsup_{x\to x_0+} s(x)$. Hence, two sequences, $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$, both contained in (c,d), can be chosen having the properties:

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x_0$;

(ii) $a_n > b_n > a_{n+1} > x_0$; and

(iii) $\lim_{n\to\infty} s(a_n) = \limsup_{x\to x_0+} s(x)$ and

$\lim_{n\to\infty} s(b_n) = \liminf_{x\to x_0+} s(x)$.

Set $\varepsilon = (q - p)/4$ and select $N_1 > 0$ such that $s(a_n) > q - \varepsilon$ and $s(b_n) < p + \varepsilon$ whenever $n \geq N_1$. By Lemma 2 there is an $N \geq N_1$ such that the boundary value problem

$$y'' = f(x,y,y')$$

$$y(b_N) = y(b_{N+1}) = \frac{p+q}{2}$$

has a solution $y(x)$ with the property that $|y(x) - (p+q)/2| \leq \varepsilon$ for $b_{N+1} \leq x \leq b_N$. Since

$$s(b_n) < p + \varepsilon = p + \frac{q-p}{4} = \frac{p+q}{2} - \frac{q-p}{4} = \frac{p+q}{2} - \varepsilon \le y(b_n)$$

for $n = N, N+1$, then $s(a_N) \le y(a_N)$ because $s(x)$ is a subfunction. However,

$$s(a_N) > q - \varepsilon = q - \frac{q-p}{4} = \frac{p+q}{2} + \frac{q-p}{4} = \frac{p+q}{2} + \varepsilon \ge y(a_N)$$

which is contradictory. Thus $s(x_0+0)$ exists, and similarly all of the other one-sided limits exist on $[c,d]$. $\square$

Theorem 1'. If $S(x)$ is a superfunction with respect to (3-1) on $[c,d]$, then the right-hand and left-hand limits, $S(x_0+0)$ and $S(x_0-0)$, exist at every x_0 in (c,d) and the appropriate one-sided limits exist at $x = c$ and $x = d$. These limits may be infinite.

Corollary 1.1. If $s(x)$ is a subfunction with respect to (3-1) on $[c,d]$, then

$$s(x_0) \le \max \{s(x_0+0), s(x_0-0)\}$$

at every x_0 in (c,d).

Proof. If either $s(x_0+0) = +\infty$ or $s(x_0-0) = -\infty$, the inequality is true. Suppose $s(x_0+0)$ and $s(x_0-0)$ are bounded from above. Then there is a real number r such that $r \ge \max \{s(x_0+0), s(x_0-0)\}$. Assume $s(x_0) > r$ and set $\varepsilon = [s(x_0) - r]/4$. Because the right-hand and left-hand limits exist at x_0, there is a $\delta_\varepsilon > 0$ for which $s(x) < r + \varepsilon$ whenever $c < x_0 - \delta_\varepsilon < x < x_0 + \delta_\varepsilon < d$. By Lemma 2 there is a $0 < \delta < \delta_\varepsilon$ such that the boundary value problem

$$y'' = f(x,y,y')$$

$$y(x_0-\delta) = y(x_0+\delta) = r + \varepsilon$$

has a solution $y(x)$ for which $|y(x) - (r+\varepsilon)| < \varepsilon$. Now

$$s(x_0) = r + 4\varepsilon > r + 2\varepsilon > y(x_0)$$

which contradicts the fact that $s(x)$ is a subfunction. Hence, $s(x_0) \leq \max \{s(x_0+0), s(x_0-0)\}$ for $c < x < d$. $\square$

<u>Corollary</u> 1.1'. If $S(x)$ is a superfunction with respect to (3-1) on $[c,d]$, then $S(x_0) \geq \min \{S(x_0+0), S(x_0-0)\}$ at every x_0 in (c,d).

<u>Corollary</u> 1.2. If $s(x)$ is a bounded subfunction with respect to (3-1) on $[c,d]$, then $s(x)$ has at most a countable number of discontinuities on $[c,d]$.

<u>Proof</u>. Let A be the set of all points in $[c,d]$ at which $s(x)$ is not continuous. For each positive integer k define

$$A_k = \{a \mid a\varepsilon A \text{ and } |s(a+0)-s(a)|>\tfrac{1}{k} \text{ or } |s(a-0)-s(a)|>\tfrac{1}{k} \text{ or } |s(a+0)-s(a-0)|>\tfrac{1}{k}\}.$$

Suppose for some $k_1 \geq 1$ that A_{k_1} has a limit point α. Because $[c,d]$ is a compact set, then $\alpha\varepsilon[c,d]$. Assume without any loss in generality that $c \le \alpha < d$. Let $\varepsilon = 1/8k_1$ and then there is a $\delta_\varepsilon > 0$ such that $|s(x) - s(\alpha+0)| < \varepsilon$ whenever $\alpha < x < \alpha + \delta_\varepsilon < d$. Since α is a limit point of A_{k_1}, say there is a point $a \varepsilon A_{k_1}$ in $(\alpha, \alpha + \delta_\varepsilon)$. Suppose $|s(a+0) - s(a-0)| > 1/k_1$. Being $s(x)$ is bounded, there is a $\delta_1 > 0$ such that $|s(x) - s(a+0)| < \varepsilon$ for $a < x \le a + \delta_1 < \alpha + \delta_\varepsilon$ and $|s(x) - s(a-0)| < \varepsilon$ for $\alpha < a - \delta_1 \le x < a$. As a result

$$
\begin{aligned}
|s(a-\delta_1) - s(\alpha+0)| &\ge |s(a-\delta_1) - s(a+\delta_1)| - |s(\alpha+0) - s(a+\delta_1)| \\
&> |[s(a-0) - \varepsilon] - [s(a+0) + \varepsilon]| - \varepsilon \\
&\ge |s(a-0) - s(a+0)| - 3\varepsilon \\
&> \frac{1}{k_1} - 3\varepsilon \\
&= 8\varepsilon - 3\varepsilon
\end{aligned}
$$

or $\varepsilon > |s(a-\delta_1) - s(\alpha+0)| > 5\varepsilon$. Similar results can be obtained if $a < \alpha$ and for the cases $|s(a+0) - s(a)| > 1/k_1$ and $|s(a-0) - s(a)| > 1/k_1$. Hence, α cannot be a limit point for A_{k_1}; equivalently, A_k is an isolated set for every $k \ge 1$. Since isolated sets are countable and $A = \bigcup_{k=1}^{\infty} A_k$, then A is a countable set. $\square$

<u>Corollary</u> 1.2'. If $S(x)$ is a bounded superfunction with respect to (3-1) on $[c,d]$, then $S(x)$ has at most a countable number of discontinuities on $[c,d]$.

Stronger results can be obtained for bounded subfunctions with respect to the differential equation

$$y'' = g(x,y) \tag{3-2}$$

as the following theorem illustrates.

<u>Theorem</u> 2. If $s(x)$ is a bounded function on $[c,d]$ and is a subfunction with respect to the solutions of the differential equation (3-2), then $s(x)$ is continuous on (c,d).

<u>Proof</u>. There exists an $M>0$ such that $|s(x)| \leq M$ for all x in $[c,d]$. Let $x_o \varepsilon (c,d)$ and suppose $s(x_o+0) > s(x_o-0)$. By Corollary 1.1 then $s(x_o) \leq s(x_o + 0)$. Assume that $s(x_o) < s(x_o+0)$. By Lemma 3 there is a $\delta(M) > 0$ such that for $[x_o,\bar{x}] \subset [c,d]$ and $|\bar{x}-x_o| \leq \delta(M)$ the boundary value problem

$$y'' = g(x,y)$$
$$y(x_o) = s(x_o), \quad y(\bar{x}) = s(\bar{x})$$

has a solution $y(x)$. Since $s(x)$ is a subfunction on $[c,d]$, then $s(x) \leq y(x)$ on $[x_o,\bar{x}]$ which implies that

$$s(x_o+0) \leq y(x_o+0) = y(x_o) = s(x_o)$$

contradicting the supposition that $s(x_o) < s(x_o+0)$. Thus $s(x_o) = s(x_o+0)$.

Now assume that $s(x_o+0) - s(x_o-0) = k > 0$. Again by Lemma 3 there is a $0 < \delta \leq \delta(M)$ such that $[x_o-\delta, x_o+\delta] \subset (c,d)$ and if $[x_1,x_2] \subset [x_o-\delta,x_o+\delta]$ the boundary value problem

$$y'' = g(x,y)$$

$$y(x_1) = s(x_1), \quad y(x_2) = s(x_2)$$

has a solution $y_1(x)$ for which $|y_1(x) - \omega(x)| < k/4$ where $\omega(x)$ is the linear function joining the points $(x_1, s(x_1))$ and $(x_2, s(x_2))$. Choose x_1, x_2 so that $x_1 < x_o < x_2$ and x_1 is close enough to x_o such that $|s(x_1) - s(x_o-0)| < k/4$ and $2M|x_1-x_o|/|x_2-x_o| < k/4$ after x_2 is chosen. Then

$$\begin{aligned} |\omega(x_o) - s(x_o-0)| &\leq |\omega(x_o) - \omega(x_1)| + |\omega(x_1) - s(x_o-0)| \\ &= \left|\frac{\omega(x_2)-\omega(x_1)}{x_2-x_1}\right| |x_o-x_1| + |s(x_1)-s(x_o-0)| \\ &< \frac{2M}{|x_2-x_o|} |x_o-x_1| + \frac{k}{4} \\ &< \frac{k}{2} \end{aligned}$$

which implies that

$$|y_1(x_o) - s(x_o-0)| \leq |y_1(x_o)-\omega(x_o)| + |\omega(x_o)-s(x_o-0)| < \frac{3k}{4}$$

Hence,

$$s(x_o) = s(x_o+0) > s(x_o-0) + \frac{3k}{4} \geq y_1(x_o)$$

which contradicts the fact that $s(x)$ is a subfunction. Similar results are obtained if it were assumed $s(x_o-0) > s(x_o+0)$. Thus $s(x)$ is a continuous function on (c,d). $\square$

<u>Theorem</u> 2'. If $S(x)$ is a bounded function on $[c,d]$ and is a superfunction with respect to (3-2), then $S(x)$ is continuous on (c,d).

The following two theorems show that subfunctions have several properties in common with convex functions.

<u>Theorem</u> 3. If $\{s_\alpha | \alpha \varepsilon A\}$ is any collection of subfunctions on $[c,d]$ bounded above at each point of $[c,d]$, then $s_o(x)$ defined by

$$s_o(x) \equiv \sup_{\alpha \varepsilon A} \{s_\alpha(x)\}$$

is a subfunction on $[c,d]$.

<u>Proof</u>. Let $y(x)$ be a solution of (3-1) on $[c,d]$, and suppose that $s_o(x_1) \leq y(x_1)$ and $s_o(x_2) \leq y(x_2)$ for $[x_1,x_2] \subset [c,d]$. Then for every α in the indexing set A, $s_\alpha(x_i) \leq y(x_i)$ for $i = 1,2$. Hence $s_\alpha(x) \leq y(x)$ for all $x \varepsilon (x_1,x_2)$ and all $\alpha \varepsilon A$. Thus $s_o(x) \leq y(x)$ on (x_1,x_2) and $s_o(x)$ is a subfunction on $[c,d]$. $\square$

<u>Theorem</u> 3'. If $\{S_\alpha | \alpha \varepsilon A\}$ is any collection of superfunctions on $[c,d]$ bounded below at each point of $[c,d]$, then $S_o(x)$ defined by

$$S_o(x) \equiv \inf_{\alpha \varepsilon A} \{S_\alpha(x)\}$$

is a superfunction on $[c,d]$.

<u>Theorem</u> 4. Let $s_1(x)$ be a subfunction on $[c,d]$ and $s_2(x)$ a subfunction on $[x_1,x_2] \subset [c,d]$. Assume further that $s_2(x_i) \leq s_1(x_i)$ for $i = 1,2$ whenever $x_i \varepsilon (c,d)$. Then

s(x) defined on [c,d] by

$$s(x) = \begin{cases} s_1(x) & \text{if } x \notin [x_1,x_2] \\ \max\{s_1(x),\ s_2(x)\} & \text{if } x \in [x_1,x_2] \end{cases}$$

is a subfunction on [c,d].

Proof. If $[x_1,x_2] = [c,d]$, Theorem 3 guarantees this result. Suppose now that there are points x_3 and x_4 in [c,d] such that $c \leq x_3 \leq x_1 < x_4 \leq x_2$ and $s(x_3) \leq y(x_3)$ and $s(x_4) \leq y(x_4)$ for y(x) a solution of (3-1) on [c,d]. Hence $s_1(x_3) \leq y(x_3)$ and $s_1(x_4) \leq s(x_4) \leq y(x_4)$ which makes $s_1(x) \leq y(x)$ on $[x_3,x_4]$. Since $s_2(x_1) \leq s_1(x_1) \leq y(x_1)$ and $s_2(x_4) \leq s(x_4) \leq y(x_4)$, then $s_2(x) \leq y(x)$ on $[x_1,x_4]$. Thus $s(x) \leq y(x)$ on $[x_3,x_4]$. Similarly, the other cases for $[x_1,x_2] \subset [c,d]$ can be shown to be true. Therefore s(x) is a subfunction on [c,d]. □

Theorem 4'. Let $S_1(x)$ be a superfunction on [c,d] and $S_2(x)$ a superfunction on $[x_1,x_2] \subset [c,d]$. Assume further that $S_2(x_i) \geq S_1(x_i)$ for i = 1,2 whenever $x_i \in (c,d)$. Then S(x) defined on [c,d] by

$$S(x) = \begin{cases} S_1(x) & \text{if } x \notin [x_1,x_2] \\ \min\{S_1(x),\ S_2(x)\} & \text{if } x \in [x_1,x_2] \end{cases}$$

is a superfunction on [c,d].

Now to investigate the differentiability of subfunctions, consider the quantities defined by

$$d^-s(x_o) \equiv \limsup_{x\to x_o+} \frac{s(x)-s(x_o-0)}{x-x_o}\ ,\quad d_-s(x_o) \equiv \liminf_{x\to x_o+} \frac{s(x)-s(x_o-0)}{x-x_o}$$

$$d^+s(x_o) \equiv \limsup_{x \to x_o+} \frac{s(x)-s(x_o+0)}{x-x_o}, \; d_+s(x) \equiv \liminf_{x \to x_o+} \frac{s(x)-s(x_o+0)}{x-x_o}$$

for any point x_o in $[c,d]$, where only one-sided limits apply at the endpoints.

Theorem 5. If $s(x)$ is a bounded subfunction on $[c,d]$, then $d^-s(x_o) = d_-s(x_o)$ for all $c < x_o \leq d$ and $d^+s(x_o) = d_+s(x_o)$ for all $c \leq x_o < d$.

Proof. Let $c \leq x_o < d$ and suppose that $d^+s(x_o) > d_+s(x_o)$. Then there is a finite number p such that $d^+s(x_o) > p > d_+s(x_o)$. Because $f(x,y,y')$ is a continuous function on its domain R_3, there is a $\delta > 0$ such that $x_o + \delta \leq d$ and the initial value problem

$$y'' = f(x,y,y')$$
$$y(x_o) = s(x_o+0), \quad y'(x_o) = p$$

has a solution $y(x)$ of class $C^{(2)}$ on $[x_o, x_o+\delta]$. Set $\varepsilon = \min\{[d^+s(x_o) - p]/4, [p-d_+s(x_o)]/4\}$ and then there is a $0 < \delta_\varepsilon < \delta$ such that

$$\left| \frac{y(x) - y(x_o)}{x-x_o} - p \right| < \varepsilon$$

or

$$y(x_o) + (p-\varepsilon)(x-x_o) < y(x) < y(x_o) + (p+\varepsilon)(x-x_o)$$

whenever $x_o < x < x_o + \delta_\varepsilon$. Since $d_+s(x_o) < p$, there are points $x_1 < x_2$ in $(x_o, x_o+\delta_\varepsilon)$ for which

$$\frac{s(x_i) - s(x_o+0)}{x_i-x_o} > p - 3\varepsilon$$

for $i = 1,2$ and a point x_{12} in (x_1,x_2) such that

$$\frac{s(x_{12}) - s(x_o+0)}{x_{12} - x_o} > p + 3\varepsilon$$

since $d^+s(x_o) > p$. This may be accomplished by choosing successively x_2 and $x_{12} \varepsilon (x_o,x_2)$ and $x_1 \varepsilon (x_o,x_{12})$. Then for $i = 1,2$

$$\begin{aligned} s(x_i) &< s(x_o+0) + (p - 3\varepsilon)(x_i-x_o) \\ &< y(x_o) + (p-\varepsilon)(x_i-x_o) \\ &< y(x_i) \end{aligned}$$

which means that $s(x) \leq y(x)$ on $[x_1,x_2]$ since $s(x)$ is a subfunction. However, $x_1 < x_{12} < x_2$ and

$$\begin{aligned} s(x_{12}) &> s(x_o+0) + (p+3\varepsilon)(x_{12}-x_o) \\ &> y(x_o) + (p+\varepsilon)(x_{12}-x_o) \\ &> y(x_{12}), \end{aligned}$$

which is a contradiction. Thus, $d^+s(x_o) = d_+s(x_o)$ and similarly, $d^-s(x_o) = d_-s(x_o)$ on the appropriate subintervals of $[c,d]$. $\square$

<u>Theorem 5'</u>. If $S(x)$ is a bounded superfunction on $[c,d]$ then $d^-S(x_o) = d_-S(x_o)$ for all $c < x_o \leq d$ and

$d^+s(x_o) = d_+s(x_o)$ for all $c \leq x_o < d$.

Corollary 5.1. If $s(x)$ is a bounded subfunction on $[c,d]$, then $s(x)$ has a finite derivative almost everywhere on $[c,d]$.

Proof. By Corollary 1.2, $s(x)$ has at most a countable number of discontinuities; and hence, only points at which $s(x)$ is continuous are considered. Let k be any rational number and let

$$E_k^+ = \{t \mid c < t < d,\ s(x) \text{ is continuous at } t, \text{ and } Ds(t+0) \geq k\}$$

where as a consequence of Theorem 5 for t in $[c,d)$ set

$$Ds(t+0) = \lim_{x \to t+} \frac{s(x)-s(t)}{x-t}$$

and for t in $(c,d]$

$$Ds(t-0) = \lim_{x \to t-} \frac{s(x)-s(t)}{x-t}$$

Suppose $t \in E_k^+$. For every $n \geq 1$ there is an $h_n > 0$ for which

$$\frac{s(x) - s(t)}{x - t} \geq k - \frac{1}{n}$$

whenever $t < x < t + h_n$ by the definition of $Ds(t+0)$. Let $\bar{h}_n$ be the maximum value of h_n satisfying this condition and such that $[t, t+\bar{h}_n] \subset [c,d]$. Let $\delta(t,n)$ denote $(t, t + \bar{h}_n/2) \cap (t, t + 1/n)$ and $\Delta_n = \{\delta(t,n) \mid t \in E_k^+\}$ for all $n \geq 1$. Then let $\Delta = \bigcup_{n=1}^{\infty} \Delta_n$. Note that by construction the elements of E_k^+ are the left-hand endpoints of the intervals in the family Δ.

Let

$$E_{k,1}^{+} = \{x \mid x\varepsilon E_k^+ \text{ and } x\not\varepsilon\delta(t,1) \text{ for all } t\varepsilon E_k^+\}.$$

Hence, if $x\varepsilon E_{k,1}^{+}$ there are no elements of E_k^+ in $\delta(x,1)$ by the definition of $E_{k,1}^{+}$. Set $\mathcal{D}_1 = \{\delta(x,1) \mid x\varepsilon E_{k,1}^{+}\}$. Suppose $\delta(x,1)$ and $\delta(t,1)$ are in $\mathcal{D}_1$ and $\delta(x,1) \cap \delta(t,1)$ is not empty. Then $x\varepsilon\delta(t,1)$ or $t\varepsilon\delta(x,1)$ which is contradictory. This implies $\mathcal{D}_1$ is a collection of disjoint open intervals on (c,d), and thus $\mathcal{D}_1$ is an enumerable collection. Since $E_{k,1}^{+}$ and $\mathcal{D}_1$ are in a one-to-one correspondence, $E_{k,1}^{+}$ is a countable set. Now denote

$$E_{k,2}^{+} = \{x \mid x\varepsilon E_k^+ \text{ and } x\not\varepsilon\delta(t,2) \text{ for all } t\varepsilon E_k^+\},$$

and then $\mathcal{D}_2 = \{\delta(x,2) \mid x\varepsilon E_{k,2}^{+}\}$ is a countable collection of disjoint open intervals. Hence, $E_{k,2}^{+}$ is an enumerable set. Similarly $E_{k,n}^{+}$ is enumerable for all $n\geq 3$, and thus the set $E_{k,0}^{+} = \bigcup_{n=1}^{\infty} E_{k,n}^{+}$ is countable.

Suppose $x_0 \varepsilon (E_k^+ - E_{k,0}^{+})$. Then for all $n\geq 1$, there is a $t\varepsilon E_k^+$ for which $x_0\varepsilon\delta(t,n)$ and thus

$$\frac{s(x_0) - s(t)}{x_0 - t} \geq k - \frac{1}{n} .$$

Hence, a sequence of points $\{t_n\}_{n=1}^{\infty}$ in E_k^+ can be found with the properties that $\{t_n\}$ converges to x_0, $t_1 \geq t_2 \geq \cdots \geq t_n \geq t_{n+1} \geq x_0$, and

$$\frac{s(x_0) - s(t_n)}{x_0 - t_n} \geq k - \frac{1}{n} .$$

Since $Ds(x_0-0)$ exists, then $Ds(x_0-0) \geq k$ for all $x_0 \varepsilon (E_k^+ - E_{k,0}^{\;+})$. Therefore, if $Ds(x+0) \geq k > Ds(x-0)$, then $x \varepsilon E_{k,0}^{\;+}$. Similarly, if $Ds(x+0) < k \leq Ds(x-0)$, then x is an element of a countable set $E_{k,0}^{\;-}$ constructed from right-hand endpoints of open intervals corresponding to elements of

$$E_k^- = \{t \mid c<t<d,\ s(x) \text{ is continuous at } t, \text{ and } Ds(t-0) \geq k\}.$$

Let $\{k_n\}$ be an enumeration of the rational numbers Q. The sets E^+ defined by

$$E^+ = \bigcup_{k_n \varepsilon Q} E_{k_n,0}^{\;+} \supset \{t \mid c<t<d, s(x) \text{ is continuous at } t, Ds(t+0) > Ds(t-0)\}$$

and E^- defined by

$$E^- = \bigcup_{k_n \varepsilon Q} E_{k_n,0}^{\;-} \supset \{t \mid c<t<d, s(x) \text{ is continuous at } t, Ds(t+0) < Ds(t-0)\}$$

are countable. Thus $Ds(x+0) = Ds(x-0)$ except on a set of measure zero.

Now let

$$E_{+\infty} = \{t \mid c<t<d,\ s(x) \text{ is continuous at } t,\ Ds(t+0) = Ds(t-0) = +\infty\}$$

and

$$E_r = \{t \mid c<t<d,\ s(x) \text{ is continuous at } t,\ Ds(t+0) = +\infty, Ds(t-0) > r\}$$

for r any rational number. Thus $E_{+\infty} \subset \bigcup_{r\epsilon Q} E_r$. For all $n \geq 1$ set

$$E_{r,n} = \{t \mid t\epsilon E_r \text{ and } \frac{s(t)-s(x)}{t-x} > r \text{ for all } t-\frac{1}{n}<x<t\},$$

and then $E_r = \bigcup_{n=1}^{\infty} E_{r,n}$. Choose any $n \geq 1$ and any rational number r, and let $\eta \epsilon E_{r,n}$. Now choose $\eta > \xi > \max\{\eta - 1/n, c\}$ and set $E_\eta = (\xi,\eta) \cap E_{r,n}$. Furthermore, let p be any number and

$$T = \{[t,\tau] \mid t\epsilon E_\eta, \ \tau<\eta, \text{ and } \frac{s(\tau)-s(t)}{\tau-t} > p\}.$$

The family T covers E_η in the sense of Vitali which implies that for every $\varepsilon>0$ there is a finite number of disjoint intervals of T, say $[t_1,\tau_1], [t_2,\tau_2],\ldots,[t_\ell,\tau_\ell]$, such that

$$|m^*(E_\eta) - \sum_{i=1}^{\ell} m([t_i,\tau_i])| < \varepsilon$$

where m is Lebesgue measure and m^* is outer measure. Hence,

$$\begin{aligned} s(\eta)-s(\xi) &= [s(\eta)-s(\tau_1)] + [s(\tau_1)-s(t_1)] + [s(t_1)-s(\tau_2)] \\ &\quad + \cdots + [s(\tau_\ell)-s(t_\ell)] + [s(t_\ell)-s(\xi)] \\ &> r(\eta-\tau_1) + p(\tau_1-t_1) + r(t_1-\tau_2) + \cdots + p(\tau_\ell-t_\ell) \\ &\quad + r(t_\ell-\xi) \\ &= r\{\eta - \xi - \sum_{i=1}^{\ell} m([t_i,\tau_i])\} + p\sum_{i=1}^{\ell} m([t_i,\tau_i]) \\ &> r[\eta - \xi - m^*(E_\eta) - \varepsilon] + p[m^*(E\ ) - \varepsilon]. \end{aligned}$$

Now suppose $\varepsilon < m^*(E_\eta)$. Since this inequality holds for all values of p, particularly for arbitrarily large values, then it must be true that $m^*(E\ )=0$. Hence, $m^*(E_{r,n}) \leq \sum_{\eta\varepsilon E_{r,n}} m^*(E_\eta)=0$ for any $n\geq 1$ and r rational. Thus $m(E_{+\infty}) = m^*(E_{+\infty}) = 0$ and similarly $m(E_{-\infty}) = 0$. Therefore a bounded subfunction has a finite derivative everywhere on [c,d] except on a set of measure zero. □

Corollary 5.1'. If S(x) is a bounded superfunction on [c,d], then S(x) has a finite derivative almost everywhere on [c,d].

For any function $\gamma(x)$ defined on [c,d] and for $x_0\varepsilon(c,d)$, denote as in Chapter II

$$\overline{D}\gamma(x_0) \equiv \limsup_{\delta\to 0} \frac{\gamma(x_0+\delta) - \gamma(x_0-\delta)}{2\delta}$$

and

$$\underline{D}\gamma(x_0) \equiv \liminf_{\delta\to 0} \frac{\gamma(x_0+\delta) - \gamma(x_0-\delta)}{2\delta}$$

Now the following necessary condition on subfunctions can be established.

Theorem 6. If s(x) is a subfunction of class $C^{(1)}$ on [c,d], then $\underline{D}s'(x) \geq f(x,s(x),s'(x))$ for $c < x < d$.

Proof. Let $x_0\varepsilon(c,d)$ and $\varepsilon>0$. Since f(x,y,z) is continuous on R_3, there is a $\rho>0$ such that the inequality $f(x,y,z) \geq f(x_0,s(x_0),s'(x_0)) - \varepsilon$ holds whenever $|x-x_0| < \rho$, $|y-s(x_0)| < \rho$, and $|z-s'(x_0)| < \rho$. Choose $\delta_0>0$ so that

$[x_o-\delta_o,\ x_o+\delta_o] \subset [c,d]$ which means there exist $M,N > 0$ such that $|s(x)| \le M$ and $|s'(x)| \le N$ for $x_o-\delta_o \le x \le x_o+\delta_o$ because $s(x) \in C^{(1)}[c,d]$. By Lemma 2 there is an $0 < \eta \le \delta(M,N) < \delta_o$ such that the boundary value problem

$$y'' = f(x,y,y')$$

$$y(x_o - \delta) = s(x_o-\delta), \quad y(x_o + \delta) = s(x_o+\delta)$$

has a solution $y(x;\delta)$ for any $0 < \delta \le \eta$ with the properties that $|y(x;\delta) - \omega(x;\delta)| < \rho/2$ and $|y'(x;\delta) - \omega'(x;\delta)| < \rho/2$, where $\omega(x;\delta)$ is the linear function passing through $(x_o-\delta, s(x_o-\delta))$ and $(x_o+\delta, s(x_o+\delta))$.

Since $s(x) \in C^{(1)}$ on $[c,d]$, there is a $\delta_1 > 0$ so that if $|x-x_o| < \delta_1$, then $|s(x) - s(x_o)| < \rho/6$ and $|s'(x)-s'(x_o)| < \rho/2$. Hence, if $\delta \le \min\{\eta,\delta_1\}$, then for $x_o-\delta < x < x_o + \delta$

$$\begin{aligned}
|y(x;\delta) - s(x_o)| &\le |y(x;\delta) - \omega(x;\delta)| + |\omega(x;\delta) - s(x_o)| \\
&< \frac{\rho}{2} + |\omega(x;\delta) - \omega(x_o-\delta;\delta)| + |\omega(x_o-\delta;\delta)-s(x_o)| \\
&\le \frac{\rho}{2} + |\omega(x_o+\delta;\delta) - \omega(x_o-\delta;\delta)| + |\omega(x_o-\delta;\delta)-s(x_o)| \\
&\le \frac{\rho}{2} + |\omega(x_o+\delta;\delta) - s(x_o)| + |s(x_o) - \omega(x_o-\delta;\delta)| \\
&\qquad + |\omega(x_o-\delta;\delta)-s(x_o)| \\
&= \frac{\rho}{2} + |s(x_o+\delta) - s(x_o)| + 2|s(x_o-\delta) - s(x_o)| \\
&< \frac{\rho}{2} + 3\left(\frac{\rho}{6}\right) \\
&= \rho.
\end{aligned}$$

An application of the Mean-Value Theorem for $x \varepsilon (x_o - \delta, x_o + \delta)$ yields

$$|y'(x;\delta) - s'(x_o)| \leq |y'(x;\delta) - \omega'(x;\delta)| + |\omega'(x;\delta) - s'(x_o)|$$

$$< \frac{\rho}{2} + \left| \frac{s(x_o + \delta) - s(x_o - \delta)}{2\delta} - s'(x_o) \right|$$

$$= \frac{\rho}{2} + |s'(\bar{x}) - s'(x_o)|$$

for some $\bar{x} \varepsilon (x_o - \delta, x_o + \delta)$ so that

$$|y'(x;\delta) - s'(x_o)| < \frac{\rho}{2} + \frac{\rho}{2} = \rho.$$

Because $s(x)$ is a subfunction on $[c,d]$ and $s(x_o \pm \delta) = y(x_o \pm \delta)$, then $s(x) \leq y(x;\delta)$ on $[x_o - \delta, x_o + \delta]$. This means that $s'(x_o - \delta) \leq y'(x_o - \delta;\delta)$ and $s'(x_o + \delta) \geq y'(x_o + \delta;\delta)$. Hence, by the Mean-Value Theorem

$$\frac{s'(x_o+\delta) - s'(x_o-\delta)}{2\delta} \geq \frac{y'(x_o+\delta;\delta) - y'(x_o-\delta;\delta)}{2\delta} = y''(\xi;\delta)$$

for some ξ in $(x_o - \delta, x_o + \delta)$. Thus

$$\frac{s'(x_o+\delta) - s'(x_o-\delta)}{2\delta} \geq f(\xi, y(\xi;\delta), y'(\xi;\delta))$$

$$\geq f(x_o, s(x_o), s'(x_o)) - \varepsilon.$$

Since this is true for all $0 < \delta < \min\{\eta, \delta_1\}$, then

$$\underline{D}s'(x_o) \equiv \liminf_{\delta \to 0} \frac{s'(x_o+\delta) - s'(x_o-\delta)}{2\delta} \geq f(x_o, s(x_o), s'(x_o)). \quad \square$$

<u>Theorem</u> 6'. If $S(x)$ is a superfunction of class $C^{(1)}$

on $[c,d]$, then $\overline{D}S'(x) \leq f(x,S(x), S'(x))$ for $c < x < d$.

The converse of Theorem 6 is not true if $f(x,y,z)$ is assumed to be continuous only. Consider the boundary value problem

$$y'' = -y$$

$$y(0) = 1, \quad y(2\pi) = 1$$

whose solution is $y(x) = \cos x$. The function $s(x) = 1/2$ satisfies the differential inequality

$$\underline{D}s'(x) = s''(x) = 0 \geq f(x,s(x), s'(x)) = -s(x) = -\frac{1}{2}$$

and also $s(\pi/3) = y(\pi/3)$ and $s(5\pi/3) = y(5\pi/3)$. However, $s(x) > y(x)$ for x in $(\pi/3, 5\pi/3)$. Therefore, $s(x)$ is not a subfunction. In this example $f(x,y,z) = -y$ which is decreasing in y. Hence, $f(x,y,z)$ must be non-decreasing in y in order for solutions of $\underline{D}s'(x) \geq f(x,s(x),s'(x))$ to be subfunctions.

Furthermore, the boundary value problem

$$y'' = 3\sqrt[3]{4}\,(y')^{2/3}$$
$$y(-1) = 1, \quad y(1) = 1$$

has the solution $y(x) = x^4$. Now $s(x) = 1$ satisfies

$$\underline{D}s'(x) = 0 \geq 3\sqrt[3]{4}\,[s'(x)]^{2/3} = 0,$$

but $s(x) > y(x)$ on $(-1,1)$ which prevents $s(x)$ from being a subfunction. Here $f(x,y,z) = 3\sqrt{4}\, z^{2/3}$ does not satisfy a Lipschitz condition on any domain containing $z=0$. Thus another sufficient condition for solutions of $\underline{D}s'(x) \geq f(x,s(x),s'(x))$ to be subfunctions with respect to (3-1) is that $f(x,y,z)$ satisfy some kind of Lipschitz condition in its third variable.

Theorem 7. Let $s(x)$ be continuous on $[c,d]$ and of class $C^{(1)}$ on (c,d). If $f(x,y,z)$ satisfies conditions A_1 and A_2 and if $\underline{D}s'(x) \geq f(x,s(x), s'(x))$ on (c,d), then $s(x)$ is a subfunction on $[c,d]$.

Proof. Let $[x_1,x_2] \subset [c,d]$ and $y(x)$ be a solution of (3-1) with $s(x_1) \leq y(x_1)$ and $s(x_2) \leq y(x_2$. Because $y(x)$ satisfies the differential inequality $\overline{D}y' = y'' \leq f(x,y,y')$ on (c,d), then $s(x) \leq y(x)$ on (x_1,x_2) by Lemma 6. Thus $s(x)$ is a subfunction on $[c,d]$. $\square$

Theorem 7'. Let $S(x)$ be continuous on $[c,d]$ and of class $C^{(1)}$ on (c,d). If $f(x,y,z)$ satisfies conditions A_1 and A_2 and if $\overline{D}S'(x) \leq f(x,S(x),S'(x))$ on (c,d), then $S(x)$ is a superfunction on $[c,d]$.

Corollary 7.1. Let $s(x)$ be continuous on $[c,d]$ and of class $C^{(1)}$ on (c,d). If $f(x,y,z)$ satisfies condition A_1 and if $\underline{D}s'(x) > f(x,s(x),s'(x))$ on (c,d), then $s(x)$ is a subfunction on $[c,d]$.

Proof. This is established using Lemma 4 in the same manner as Theorem 7. $\square$

Corollary 7.1'. Let $S(x)$ be continuous on $[c,d]$ and of class $C^{(1)}$ on (c,d). If $f(x,y,z)$ satisfies condition A_1 and if $\overline{D}S'(x) < f(x,S(x),S'(x))$ on (c,d), then $S(x)$ is a superfunction on $[c,d]$.

Corollary 7.2. Let $f(x,y,z')$ satisfy conditions A_1 and A_2. Then the solution of the boundary value problem

$$y'' = f(x,y,y')$$
$$y(x_1) = y_1, \quad y(x_2) = y_2$$

for $[x_1,x_2] \subset [c,d]$, if it exists, is unique.

Proof. Suppose $\tilde{y}(x)$ and $y(x)$ are solutions of this boundary value problem. Being $\tilde{y}(x_1) = y(x_1)$ and $\tilde{y}(x_2) = y(x_2)$ and because $y(x)$ is a subfunction and $\tilde{y}(x)$ is a superfunction, then $y(x) \leq \tilde{y}(x)$ on $[x_1,x_2]$ by Lemma 6. Similarly $y(x)$ is a superfunction and $\tilde{y}(x)$ is a subfunction which implies $y(x) \geq \tilde{y}(x)$ on $[x_1,x_2]$. Therefore $y(x) = \tilde{y}(x)$ on $[x_1,x_2]$. $\square$

To illustrate these results consider the differential equation $y'' = -e^{y'}$ on any finite interval $[c,d]$ where $c > 0$. The ordinary differential equation has solutions $y(x) = x - x \ell n\, x + k$ for arbitrary constant k. Since $-e^z$ is a strictly decreasing function and $|(-e^z)'| = e^z$ is bounded on every compact interval I, the Mean-Value Theorem gives that for any $z_1, z_2 \in I$

$$|(-e^{z_1}) - (-e^{z_2})| \leq M\,|z_2 - z_1|$$

where $M = \max_{z\varepsilon I} |(-e^z)'|$. Hence, $f(x,y,z) = -e^z$ satisfies conditions A_1 and A_2. One family of subfunctions for this differential equation is the set of parabolas $\{a_o x^2 + a_1 x + a_2\}$ for $a_o \geq 0$. This is true because if $s(x) = a_o x^2 + a_1 x + a_2$,

$$\underline{D}s'(x) = 2a_o \geq -e^{s'(x)} = -e^{2a_o x + a_1}.$$

In particular, all straight lines are subfunctions.

Similarly the differential equation $y'' = e^{-y'}$, defined on intervals $[c,d]$ where $c > 0$, has solutions $y(x) = x \ln x - x + k_1$ for constant k_1. One family of superfunctions is the set of parabolas of the form $-a_o x^2 + a_1 x + a_2$ for $a_o \geq 0$.

Next a relationship between subfunctions and superfunctions, when they both exist, is determined.

Theorem 8. Let $s(x)$ be a continuous subfunction and $S(x)$ a continuous superfunction on $[c,d]$ with $s(c) \leq S(c)$ and $s(d) \leq S(d)$. Assume that at least one of $s(x)$, $S(x)$ is of class $C^{(1)}$ on (c,d); to be specific, suppose $S(x)$ is. If $f(x,y,z)$ satisfies A_1 and A_2 or $f(x,y,z)$ satisfies A_1 and $\overline{D}S'(x) < f(x,S(x),S'(x))$, then $s(x) \leq S(x)$ on $[c,d]$.

Proof. Assume that $s(x) > S(x)$ for $x \varepsilon [x_1,x_2] \subset (c,d)$ and let $M = \max_{c \leq x \leq d} \{s(x) - S(x)\} > 0$. Further, assume that x_1, x_2 are chosen so that $s(x_i) - S(x_i) \leq M/4$ for $i=1,2$ and $s(x) - S(x) = M$ for some point $x \varepsilon (x_1,x_2)$. Suppose $x_o \varepsilon (x_1,x_2)$ such that $s(x_o) - S(x_o) = M$ and $s(x) - S(x) < M$

for $x_o < x \leq x_2$. Since $S(x)$ is of class $C^{(1)}$ on $[x_1, x_2]$, then $|S'(x)|$ has a maximum value N on $[x_1, x_2]$. Hence by Lemma 1 there is a $\delta = \delta(M,N) > 0$ such that $x_1 < x_o - \delta < x_o + \delta < x_2$ and the boundary value problem

$$y'' = f(x,y,y')$$

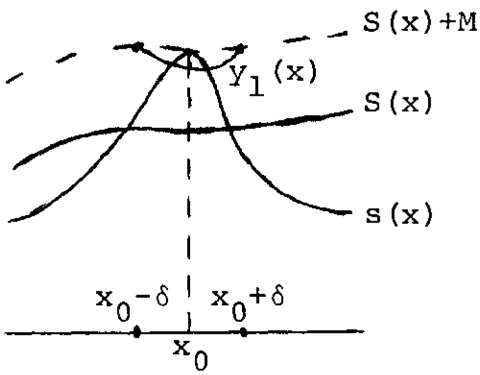
$$y(x_o - \delta) = S(x_o - \delta) + M, \quad y(x_o + \delta) = S(x_o + \delta) + M$$

has a solution $y_1(x)$ of class $C^{(2)}$ on $[x_o - \delta,\ x_o + \delta]$.

Suppose $f(x,y,z)$ satisfies A_1 and $\bar{D}S'(x) < f(x,S(x),S'(x))$ on (c,d). By condition A_1

$$\begin{aligned}
\bar{D}\ \{[S(x) + M]'\} &= \bar{D}S'(x) \\
&< f(x,S(x),S'(x)) \\
&\leq f(x,S(x) + M, S'(x)) \\
&= f(x,S(x) + M,\ [S(x) + M]'\}.
\end{aligned}$$

Because $y_1(x)$ satisfies the differential inequality $y_1''(x) \geq f(x,y_1(x),\ y_1'(x))$, Lemma 4 gives that $y_1(x) < S(x) + M$ on $(x_o - \delta,\ x_o + \delta)$.



Now

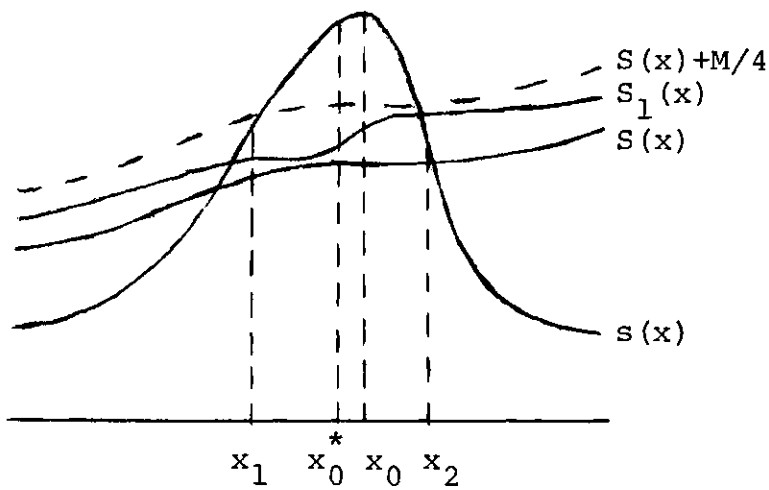
$$s(x_0 \pm \delta) \leq S(x_0 \pm \delta) + M = y_1(x_0 \pm \delta)$$

and hence $s(x) \leq y_1(x)$ on $(x_0-\delta, x_0+\delta)$ being $s(x)$ is a subfunction. This means that in particular,

$$s(x_0) \leq y_1(x_0) < S(x_0) + M = s(x_0)$$

which is contradictory. Thus $s(x) \leq S(x)$ on $[c,d]$.

Now suppose instead that $f(x,y,z)$ satisfies conditions A_1 and A_2. Since $S(x) \varepsilon C^{(1)}(c,d)$, there is a function $S_1(x)$ in $C^{(1)}[x_1,x_2]$ by Lemma 5 such that $\overline{D}S_1'(x) < f(x,S_1(x),S_1'(x))$ on (x_1,x_2) and $S(x) \leq S_1(x) \leq S(x) + M/4$ on $[x_1,x_2]$.



At the endpoints

$$s(x_i) - S_1(x_i) \leq s(x_i) - S(x_i) \leq \frac{M}{4}$$

for i=1,2. Because

$$s(x_0) - S_1(x_0) \geq s(x_0) - [S(x_0) + \frac{M}{4}]$$
$$= [s(x_0) - S(x_0)] - \frac{M}{4}$$
$$= \frac{3M}{4} ,$$

then $s(x) - S_1(x)$ has a maximum value $M_1 \geq 3M/4$ at some point x_0^* in (x_1,x_2) and $s(x) - S_1(x) < M_1$ for $x_0^* < x \leq x_2$. Replacing $S(x)$ by $S_1(x)$ and M by M_1 in the argument for the previous case again results in a contradiction. Thus $s(x) \leq S(x)$ for $c \leq x \leq d$. □

This same result also holds when $s(x) \varepsilon C^{(1)}(c,d)$ and when $f(x,y,z)$ satisfies A_1 and $\underline{D}s'(x) > f(x,s(x),s'(x))$ on (c,d).

Corollary 8.1. Let $B>0$ be a constant and assume that $f(x,y,z)$ satisfies A_1 and A_2. If $s(x)$ is a continuous subfunction on $[c,d]$, then $s(x) - B$ is also. Similarly, if $S(x)$ is a continuous superfunction on $[c,d]$, $S(x) + B$ is also.

Proof. Let $s(x)$ be a continuous subfunction on $[c,d]$. For $c \leq x_1 < x_2 \leq d$ let $y(x)$ be a solution of (3-1) with $y(x_i) \geq s(x_i) - B$ for $i=1,2$. On $[x_1,x_2]$ set $y_1(x) = y(x) + B$. Using condition A_1

$$\overline{D}y_1'(x) = y''(x) = f(x,y(x),y'(x)) \leq f(x,y_1(x),y_1'(x)).$$

Hence, $y_1(x)$ is a continuous superfunction on $[x_1,x_2]$ of class $C^{(1)}$ on (x_1,x_2). Since $y_1(x_i) = y(x_i) + B \geq s(x_i)$ for $i = 1,2$, Theorem 8 then gives that $y_1(x) \geq s(x)$ on $[x_1,x_2]$. This means that $y(x) \geq s(x) - B$ for $x_1 \leq x \leq x_2$ which shows that $s(x) - B$ is a subfunction. Similarly $S(x) + B$ can be shown to be a continuous superfunction. $\square$

So far some of the properties of sub- and superfunctions and sufficient conditions for functions to be sub- and superfunctions have been established. Now conditions insuring the existence of sub- and superfunctions are obtained.

Theorem 9. Let $f(x,y,y')$ satisfy A_1 and A_2 and there is a constant $K > 0$ such that

$$|f(x,0,y') - f(x,0,0)| \leq K|y'| \tag{3-3}$$

for all $c \leq x \leq d$ and $|y'| < \infty$. Then there exist a subfunction $s(x)$ of class $C^{(2)}$ on $[c,d]$ and a superfunction $S(x)$ of class $C^{(2)}$ on $[c,d]$ with respect to (3-1).

Proof. Because $f(x,y,z)$ is continuous on the region R_3, $|f(x,0,0)|$ has a maximum value on $[c,d]$ and let $\mu \equiv \max_{c \leq x \leq d} |f(x,0,0)|$. Set

$$\varepsilon_1 = -\frac{\mu}{K+1} - 1 < 0$$

and for finite constants σ_1, σ_2

$$N_1 = \max\left\{\frac{\sigma_1}{\varepsilon_1} + 1, \frac{\sigma_2}{\varepsilon_1} + e^{(K+1)(d-c)}, e^{(K+1)(d-c)}\right\}.$$

Now define

$$s(x) = \varepsilon_1 \left[N_1 - e^{(K+1)(x-c)}\right]$$

for $c \leq x \leq d$. Since $f(x,y,z)$ satisfies conditions A_1 and A_2, it suffices to show that $s(x)$ satisfies the differential inequality $\underline{D}s'(x) \geq f(x,s(x),s'(x))$ on (c,d) in order to establish that $s(x)$ is a subfunction using Theorem 7 as $s(x)$ is clearly continuous on $[c,d]$ and of class $C^{(1)}$ on (c,d). In fact, this function is of class $C^{(2)}$ on $[c,d]$. For any $x\varepsilon(c,d)$

$$\begin{aligned}
\underline{D}s'(x) &= D_x^{\,2}\ \{\varepsilon_1\ [N_1 - e^{(K+1)(x-c)}]\} \\
&= D_x\ [-\varepsilon_1 (K+1) e^{(K+1)(x-c)}] \\
&= (K+1)s'(x) \\
&= K\ s'(x) + s'(x) \\
&\geq f(x,0,s'(x)) - f(x,0,0) + s'(x)
\end{aligned}$$

since

$$\begin{aligned}
s'(x) &= -\varepsilon_1\ (K+1)e^{(K+1)(x-c)} \\
&= (\frac{\mu}{K+1} + 1)(K+1)e^{(K+1)(x-c)} \\
&= (\ \mu + K + 1)e^{(K+1)(x-c)} \\
&> 0
\end{aligned}$$

which means (3-3) becomes $-Ks' \leq f(x,0,s') - f(x,0,0) \leq Ks'$. Using condition A_1 and the fact that

$$s(x) = \varepsilon_1 [N_1 - e^{(K+1)(x-c)}] \leq (-\frac{\mu}{K+1} - 1)[e^{(K+1)(d-c)} - e^{(K+1)(x-c)}] < 0$$

for $c < x < d$ makes

$$\begin{aligned} \underline{D}s'(x) &\geq f(x,0,s'(x)) - f(x,0,0) + s'(x) \\ &\geq f(x,s(x),s'(x)) - \mu + s'(x) \\ &= f(x,s(x),s'(x)) - \mu + (\mu + K + 1)e^{(K+1)(x-c)} \\ &\geq f(x,s(x),s'(x)) - \mu + (\mu + K + 1) \\ &\geq f(x,s(x),s'(x)). \end{aligned}$$

Hence, $s(x)$ is a subfunction on $[c,d]$.

Similarly letting $\varepsilon_2 = \frac{\mu}{K+1} + 1$ and

$$N_2 = \max \{\frac{\sigma_1}{\varepsilon_2} + 1, \frac{\sigma_2}{\varepsilon_2} + e^{(K+1)(d-c)}, e^{(K+1)(d-c)}\},$$

the function

$$S(x) = \varepsilon_2 [N_2 - e^{(K+1)(x-c)}]$$

can be shown to be a superfunction of class $C^{(2)}$ on $[c,d]$. $\square$

Observe that at the endpoints of $[c,d]$ this constructed subfunction satisfies

$$s(c) = \varepsilon_1(N_1 - 1) \leq \varepsilon_1(\frac{\sigma_1}{\varepsilon_1}) = \sigma_1$$

and

$$s(d) = \varepsilon_1[N_1-e^{(K+1)(d-c)}] \leq \varepsilon_1(\frac{\sigma_2}{\varepsilon_1}) = \sigma_2$$

since $\varepsilon_1 < 0$. Also $S(c) \geq \sigma_1$ and $S(d) \geq \sigma_2$. Thus there exist a subfunction and a superfunction of class $C^{(2)}$ on [c,d] to the boundary value problem

$$y'' = f(x,y,y')$$

$$y(c) = \sigma_1, \; y(d) = \sigma_2.$$

A less restrictive condition than (3-3) on $f(x,y,y')$ can be used at the expense of interval length to get a similar existence result.

Theorem 10. Let $f(x,y,y')$ satisfy A_1 and A_2 and for some $v > 0$ there is $K > 0$ such that

$$|f(x,0,y') - f(x,0,0) \leq K |y'|^{1+v} \tag{3-4}$$

for all $c \leq x \leq d$ and $|y'| < \infty$. For any subinterval $[x_1,x_2]$ of $[c,d]$ such that

$$x_2 - x_1 < \frac{1}{v(K + \mu)}, \tag{3-5}$$

there exist a subfunction $s(x)$ and a superfunction $S(x)$ of class $C^{(2)}$ on $[x_1,x_2]$ with respect to the boundary value problem

$$y'' = f(x,y,y')$$

$$y(x_1) = \sigma_1, \; y(x_2) = \sigma_2.$$

Proof. The existence of a subfunction is established by investigating solutions $\phi(x)$ of the differential equation

$$z''(x) = K_1 \, [z'(x)]^{1+v} \tag{3-6}$$

where $K_1 = K + \mu$. The cases $v = 1$ and $v \neq 1$ are handled first.

For $v = 1$, consider the function

$$\phi_1(x) = \frac{1}{K_1} \ln \frac{1 - K_1(x_2 - x_1)}{1 - K_1(x - x_1)} + \xi_1$$

defined for $x_1 \leq x \leq x_2$ where

$$\xi_1 = \min \{0, \sigma_2, \sigma_1 - \frac{1}{K_1} \ln [1 - K_1(x_2 - x_1]\}.$$

This function is well defined on $[x_1, x_2]$ because by (3-5)

$$1 \geq 1 + K_1(x - x_1) \geq 1 - K_1(x_2 - x_1) = 1 - (K+\mu)(x_2 - x_1) > 0.$$

Observe also that $\xi_1 \leq 0$ and $1 - K_1(x_2 - x_1) \leq 1 - K_1(x - x_1)$ which means that $\phi_1(x) \leq 0$. Now

$$\phi_1'(x) = \frac{K_1}{1 - K_1(x - x_1)} \geq 1$$

since $1 \geq 1 - K_1(x - x_1) > 0$, and then

$$\phi_1''(x) = \frac{K_1}{[1 - K_1(x - x_1)]^2} = K_1[\phi_1'(x)]^2$$

which verifies that $\phi_1(x)$ is a solution of (3-6) of class $C^{(2)}$ on $[x_1, x_2]$.

For the case $v \neq 1$, let

$$\phi_v(x) = \frac{1}{K_1(1-v)}\{[1 - vK_1(x - x_1)]^{(v-1)/v} - [1 - vK_1(x_2 - x_1)]^{(v-1)/v}\} + \xi_v$$

for $x_1 \le x \le x_2$ where

$$\xi_v = \min \{0, \sigma_2, \sigma_1 - \frac{1}{K_1(1-v)} \{1 - [1 - vK_1(x_2-x_1)]^{(v-1)/v}\}\}.$$

By (3-5) again observe that on $[x_1,x_2]$

$$1 \ge 1 - vK_1(x-x_1) \ge 1 - vK_1(x_2-x_1) = 1 - v(K+\mu)(x_2-x_1) > 0.$$

Now for $x_1 \le x \le x_2$ rewrite $\phi_v(x)$ in the form

$$\phi_v(x) = \frac{1}{K_1(1-v)}\left\{1 - \left[\frac{1 - vK_1(x_2-x_1)}{1 - vK_1(x-x_1)}\right]^{(v-1)/v}\right\} + \xi_v$$

$$\le \frac{1}{K_1(1-v)}\left\{1 - \left[\frac{1 - vK_1(x_2-x_1)}{1 - vK_1(x-x_1)}\right]^{(v-1)/v}\right\}$$

$$= - \frac{1}{K_1(v-1)} \{1 - [r(x)]^{(v-1)/v}\}$$

using the substitution

$$r(x) = \frac{1 - vK_1(x_2-x_1)}{1 - vK_1(x-x_1)} .$$

Since $1 - vK_1(x-x_1) \ge 1 - vK_1(x_2-x_1) > 0$, then $0 < r(x) \le 1$. If $v > 1$, this makes $v - 1 > 0$ and $0 < [r(x)]^{(v-1)/v} \le 1$. If $0 < v < 1$, then $v - 1 < 0$ and $[r(x)]^{(v-1)/v} \ge 1$. In both situations

$$- \frac{1}{K_1(v-1)} \{1 - [r(x)]^{(v-1)/v}\} \le 0$$

which implies $\phi_v(x) \le 0$ on $[x_1,x_2]$. Because $1 \ge 1-vK_1(x-x_1)>0$,

$$\phi_v'(x) = [1 - vK_1(x-x_1)]^{-1/v} \geq 1.$$

Further,

$$\phi_v''(x) = K_1[1 - vK_1(x-x_1)]^{(-1/v)-1}$$

$$= K_1[1 - vK_1(x-x_1)]^{(-1/v)(1+v)}$$

$$= K_1[\phi_v'(x)]^{1+v}$$

which shows that $\phi_v(x)$ is also a solution of (3-6) of class $C^{(2)}$ on $[x_1,x_2]$.

Now denote

$$\phi(x) = \begin{cases} \phi_1(x) & \text{if } v = 1 \\ \phi_v(x) & \text{if } v \neq 1. \end{cases}$$

Then $\phi(x) \leq 0$ and $\phi'(x) \geq 1$ on $[x_1,x_2]$ and $\phi(x)$ is a solution of (3-6). Using these facts and (3-4) implies that

$$\underline{D}\phi'(x) = K_1[\phi'(x)]^{1+v}$$

$$= K[\phi'(x)]^{1+v} + \mu[\phi'(x)]^{1+v}$$

$$\geq f(x,0,\phi'(x)) - f(x,0,0) + \mu[\phi'(x)]^{1+v}$$

$$\geq f(x,0,\phi'(x)) - f(x,0,0) + \mu$$

$$\geq f(x,0,\phi'(x))$$

since $K_1 = K + \mu$ and $\mu = \max\limits_{c \leq x \leq d} |f(x,0,0)|$. By condition A_1,

$\underline{D}\phi'(x) \geq f(x,\phi(x),\phi'(x))$ which establishes that $\phi(x)$ is a subfunction by Theorem 7.

In both of the cases $v = 1$ and $v \neq 1$, $\phi(x_1) \leq \sigma_1$ and $\phi(x_2) \leq \sigma_2$. Thus $\phi(x)$ is a subfunction of class $C^{(2)}$ on $[x_1,x_2]$ with respect to the boundary value problem

$$y'' = f(x,y,y')$$

$$y(x_1) = \sigma_1,\ y(x_2) = \sigma_2.$$

On the other hand the solutions $\Psi(x)$ of $z''(x) = -K_1[z'(x)]^{1+v}$ given by

$$\Psi_1(x) = \frac{1}{K_1}\ \ell n\ \frac{1 + K_1(x-x_2)}{1 + K_1(x_1-x_2)} + \eta_1$$

where

$$\eta_1 = \max\ \{0,\ \sigma_1,\ \sigma_2 + \frac{1}{K_1}\ \ell n\ [K_1(x_1-x_2) + 1]\}$$

for $v = 1$ and for $v \neq 1$ by

$$\Psi_v(x) = \frac{1}{K_1(v-1)}\ \{[1+vK_1(x-x_2)]^{(v-1)/v} - [1+vK_1(x_1-x_2)]^{(v-1)/v}\} + \eta_v$$

where

$$\eta_v = \max\ \{0,\ \sigma_1,\ \sigma_2 + \frac{1}{K_1(v-1)}\ \{[1 + vK_1(x_1-x_2)]^{(v-1)/v} - 1\}\}$$

have the properties that $\Psi(x) \geq 0$ and $\Psi'(x) \geq 1$ on $[x_1,x_2]$ and $\overline{D}\Psi'(x) \leq f(x,\Psi(x),\Psi'(x))$. At the endpoints $\Psi(x_1) \geq \sigma_1$ and $\Psi(x_2) \geq \sigma_2$. These functions of class $C^{(2)}$ on $[x_1,x_2]$ are superfunctions with respect to this boundary value problem. $\square$

Another slightly less restrictive existence condition than Theorem 9 which is independent of interval length is provided by the next result.

Theorem 11. Let $f(x,y,y')$ satisfy A_1 and A_2 and suppose for some $K > 0$

$$|f(x,0,y') - f(x,0,0)| \leq K(|y'|\ell n\ |y'| + 1) \qquad (3\text{-}7)$$

for all $x \varepsilon [c,d]$ and $|y'| < \infty$. Then there exist a subfunction $s(x)$ and a superfunction $S(x)$ of class $C^{(2)}$ on $[c,d]$ with respect to the boundary value problem

$$y'' = f(x,y,y')$$

$$y(c) = \sigma_1,\ y(d) = \sigma_2.$$

Proof. Choose $K' \leq -(2K+\mu) < 0$ and consider the differential equation

$$y''(x) = -K'y'(x)\ \ell n\ |y'(x)|.$$

Suppose $\phi(x)$ is a solution and $\phi'(x) > 1$ on $[c,d]$. Then

$$\frac{\phi''(x)}{\phi'(x)\ \ell n\ \phi'(x)} = -K'.$$

Upon integration,

$$\ell n[\ell n \phi'(x)] = -K'x + c_1$$

where c_1 is a constant of integration.

Hence, $\ell n\phi'(x) = e^{-K'x+c_1}$ and $\phi'(x) = e^{e^{-K'x+c_1}} > 1$ on $[c,d]$. Further integration gives

$$\phi(x) = \int_c^x e^{e^{-K't+c_1}} dt + c_2$$

Let $s(x)$ be the specific solution corresponding to $c_1 = K'c$ and

$$c_2 = \min \{ - \int_c^d e^{e^{-K'(t-c)}} dt, \sigma_1, \sigma_2 - \int_c^d e^{e^{-K'(t-c)}} dt\}.$$

This solution has the properties $s(x) \leq 0$ and $s'(x) = e^{e^{-K'(x-c)}} \geq e$ on $[c,d]$ and is of class $C^{(2)}$ on $[c,d]$. Hence,

$$\begin{aligned}
\underline{D}s'(x) &= -K's'(x)\ \ell n\ s'(x) \\
&= (2K+\mu)s'(x)\ \ell ns'(x) \\
&= Ks'(x)\ \ell ns'(x) + (K+\mu)s'(x)\ \ell ns'(x) \\
&\geq Ks'(x)\ \ell ns'(x) + K + \mu \\
&= K[s'(x)\ \ell ns'(x) + 1] + \mu.
\end{aligned}$$

Then by (3-7) and condition A_1,

$$\underline{D}s'(x) \geq f(x,0,s'(x)) - f(x,0,0) + \mu \geq f(x,s(x),s'(x))$$

which implies $s(x)$ is a subfunction using Theorem 7. At the endpoints

$$s(c) = c_2 \le \sigma_1$$

and

$$s(d) = \int_c^d e^{e^{-K'(t-c)}}\,dt + c_2 \le \sigma_2.$$

Thus s(x) is a subfunction with respect to the give boundary value problem. Note that if $K' < -(2K+\mu)$, then s(x) satisfies the strict differential inequality $\underline{D}s'(x) > f(x,s(x),s'(x))$ which only requires $f(x,y,y')$ to satisfy A_1 to insure s(x) is a subfunction by Corollary 7.1.

Similarly, a solution S(x) of class $C^{(2)}$ on [c,d] of $y''(x) = K'y'(x)\,\ln|\,y'(x)|$ exists with the properties that $S(x) \ge 0$, $S'(x) \ge e$, $S(c) \ge \sigma_1$, and $S(d) \ge \sigma_2$. This solution can be shown to be a superfunction with respect to the boundary value problem. □

Existence conditions for subfunctions and superfunctions with respect to the differential equation $y'' = g(x,y)$ are much more general than those for $y'' = f(x,y,y')$ as this concluding result shows.

<u>Theorem</u> 12. If g(x,y) satisfies A_1, then there exist a subfunction s(x) of class $C^{(2)}$ on [c,d] satisfying $s''(x) > g(x,s(x))$ and a superfunction S(x) of class $C^{(2)}$ on [c,d] satisfying $S(x) < g(x,S(x))$ on [c,d] with respect to the boundary value problem

$$y'' = g(x,y)$$
$$y(c) = \sigma_1, \quad y(d) = \sigma_2.$$

Proof. Let $\omega(x)$ be the linear function with $\omega(c) = \sigma_1$ and $\omega(d) = \sigma_2$. Consider the boundary value problem

$$u''(x) = |g(x, \omega(x))| + 1$$

$$u(c) = u(d) = 0.$$

Since $g(x,\omega(x))$ is continuous and bounded on the compact interval $[c,d]$,

$$\int_c^x u''(t)dt = \int_c^x [|g(t,\omega(t))| + 1]\, dt$$

for $c \leq x \leq d$ or

$$u'(x) = u'(c) + \int_c^x [|g(t,\omega(t))| + 1]\, dt.$$

Integrating further and using the boundary condition $u(c) = 0$,

$$u(x) = u'(c)(x-c) + \int_c^x \int_c^\tau [|g(t,\omega(t))| + 1]\, dtd\tau.$$

Use of the other boundary condition, $u(d) = 0$, shows that

$$u'(c) = -\frac{1}{d-c}\int_c^d\int_c^\tau [|g(t,\omega(t))| + 1]\, dtd\tau.$$

Hence,

$$u(x) = -\frac{x-c}{d-c}\int_c^d\int_c^\tau [|g(t,\omega(t))|+1]dtd\tau + \int_c^x\int_c^\tau [|g(t,\omega(t))|+1]dtd\tau$$

is a solution of class $C^{(2)}$ on $[c,d]$ of this boundary value problem.

Now set $s(x) = u(x) + \omega(x)$. Then on $[c,d]$

$\underline{D}s'(x) = u''(x)+\omega''(x) = u''(x) = |g(x,\omega(x))|+1 > g(x,\omega(x))$.

Since $u''(x) = |g(x,\omega(x))|+1>0$ and $u(c)=u(d)=0$, then $u(x)$ is strictly concave up or $u(x)<0$ on $[c,d]$. This results in

$$\underline{D}s'(x) > g(x,\omega(x)) \geq g(x,\omega(x)+u(x)) = g(x,s(x))$$

using A_1. Thus by Theorem 7 $s(x)$ is a subfunction. Also $s(x)$ is of class $C^{(2)}$ on $[c,d]$. Furthermore, $s(c)=u(c)+\omega(c)=\sigma_1$ and $s(d)=u(d)+\omega(d)=\sigma_2$.

Similarly, if $v(x)$ is the solution of the boundary value problem

$$v''(x) = -|g(x,\omega(x))| - 1$$
$$v(c) = v(d) = 0,$$

then $S(x) = v(x) + \omega(x)$ is a superfunction of class $C^{(2)}[c,d]$ and $S(c) = \sigma_1$ and $S(d) = \sigma_2$. $\square$

CHAPTER IV

A GENERALIZED SOLUTION OF THE BOUNDARY VALUE PROBLEM

Now that several conditions ensuring the existence of subfunctons and superfunctions and some of their properties have been established, these are incorporated in the manner of Perron in the construction of a solution, called the generalized solution, to the boundary value problem

$$\begin{aligned} y'' &= f(x,y,y') \\ y(a) &= \alpha, \quad y(b) = \beta \end{aligned} \tag{4-1}$$

for the compact interval $[a,b]$ and finite real numbers α and β. Before this is done, several more items of groundwork must be briefly specified.

Definition 3. The function $\phi(x)$ is said to be an underfunction with respect to the boundary value problem (4-1) in case $\phi(x)$ is a subfunction on $[a,b]$ with $\phi(a) \leq \alpha$ and $\phi(b) \leq \beta$.

Definition 4. The function $\psi(x)$ is said to be an overfunction with respect to the boundary value problem (4-1) in case $\psi(x)$ is a superfunction on $[a,b]$ with $\psi(a) \geq \alpha$ and $\psi(b) \geq \beta$.

The construction of this generalized solution initially depends upon the existence of underfunctions and overfunctions having certain continuity properties guaranteed by Theorem 9. From this result the new assumption on $f(x,y,y')$

A_3: For all x in [a,b] and $|y'| < \infty$, there is a $K > 0$ such that

$$|f(x,0,y') - f(x,0,0)| \leq K|y'|.$$

along with A_1 and A_2 ensure the existence of an underfunction and an overfunction with respect to (4-1) of class $C^{(2)}$ on [a,b]. With this basis the generalized solution H(x) on [a,b] may now be specified.

Definition 5. Let $\{\phi\}$ represent the collection of all underfunctions with respect to the boundary value problem (4-1) which are continuous on [a,b]. Then define

$$H(x) \equiv \sup \{\phi(x) |\ \phi \varepsilon \{\phi\}\}$$

for each $x \varepsilon [a,b]$.

In analogy with a method due to Perron for first order initial value problems, several aspects of the nature of H(x) and its subordinate role to the solutions of $y'' = f(x,y,y')$ are investigated.

Theorem 13. If $f(x,y,y')$ satisfies conditions A_1, A_2, and A_3, then H(x) is a bounded subfunction on [a,b].

Proof. By A_i, i=1,2,3, the collection $\{\phi\}$ is nonempty since there exists an underfunction $\phi_0(x)$ of class $C^{(2)}[a,b]$. There also exists an overfunction $\psi_0(x)$ of class $C^{(2)}[a,b]$ and $\phi_0(a) \leq \alpha \leq \psi_0(a)$ and $\phi_0(b) \leq \beta \leq \psi_0(b)$. Theorem 8 ensures that $\phi_0(x) \leq \psi_0(x)$ on [a,b]. Similarly for any $\phi \varepsilon \{\phi\}$, on [a,b], $\phi(x) \leq \psi_0(x)$ which implies that $\phi_0(x) \leq H(x) \leq \psi_0(x)$ on [a,b]. The continuity of $\phi_0(x)$ and $\psi_0(x)$ then make H(x) a bounded function on [a,b]. By Theorem 3 H(x) is a subfunction. □

Theorem 14. If f(x,y,y') satisfies A_1, A_2, and A_3, then H(x) is a superfunction on [a,b].

Proof. Assume H(x) is not a superfunction. Then there is an interval $[x_1,x_2] \subset [a,b]$ and a solution y(x) of the differential equation $y'' = f(x,y,y')$ on $[x_1,x_2]$ such that $H(x_i) \geq y(x_i)$ for i=1,2, but $H(x) < y(x)$ for some x in (x_1,x_2). Let $x_1<x_0<x_2$ be such that $y(x_0) - H(x_0) = m > 0$. By the definition of H(x), there are continuous underfunctions $\phi_1(x)$ and $\phi_2(x)$ such that $H(x_1)- \phi_1(x_1)\leq m/4$ and $H(x_2)-\phi_2(x_2)\leq m/4$. By Corollary 8.1, y(x) - m/2 is also a subfunction on $[x_1,x_2]$. Using Theorems 3 and 4, define the continuous underfunction $\phi_3(x)$ on [a,b] by

$$\phi_3(x) = \begin{cases} \max\{\phi_1(x),\phi_2(x)\} & \text{if } x\varepsilon[a,b]-[x_1,x_2] \\ \max\{\phi_1(x),\phi_2(x),y(x)-m/2\} & \text{if } x\varepsilon[x_1,x_2] \end{cases}$$

since for i=1,2

$$y(x_i)-\frac{m}{2} \leq H(x_i)-\frac{m}{2} \leq [\phi_i(x_i)+\frac{m}{4}] - \frac{m}{2} \leq \max\{\phi_1(x_i),\phi_2(x_i)\}.$$

However, $\phi_3\varepsilon\{\phi\}$ and

$$\phi_3(x_0) \geq y(x_0) - \frac{m}{2} = [H(x_0)+m] - \frac{m}{2} > H(x_0)$$

which contradicts the definition of H(x). Thus H(x) must also be a superfunction on [a,b]. □

<u>Corollary</u> 14.1. For each $x \varepsilon (a,b)$

$$H(x) = \min \{H(x+0), H(x-0)\}.$$

<u>Proof</u>. Because $H(x)$ is a superfunction, Corollary 1.1' shows that $H(x) \geq \min \{H(x+0), H(x-0)\}$. Since each $\phi \varepsilon \{\phi\}$ is continuous on [a,b], each $\phi(x)$ is also lower semicontinuous making $H(x)$, the supremum of lower semicontinuous functions, lower semicontinuous on [a,b]. Suppose for some $x_0 \varepsilon (a,b)$ that $H(x_0) > \min \{H(x_0+0), H(x_0-0)\}$ and to be specific assume $H(x_0+0) > H(x_0-0)$. Let $\varepsilon = H(x_0) - H(x_0-0) > 0$. There exists a $\delta_\varepsilon > 0$ such that $H(x) - H(x_0-0) < \varepsilon/4$ or $H(x_0-0) > H(x_0) - \varepsilon/4$ for all $a < x_0 - \delta_\varepsilon < x < x_0$. By the lower semicontinuity of $H(x)$ at x_0, there is a $\delta > 0$ such that $H(x) > H(x_0) - \varepsilon/4$ for all $x \varepsilon (x_0 - \delta, x_0 + \delta) \subset (a,b)$. Now set $\delta_1 = \min \{\delta_\varepsilon, \delta\}$. Then for any $x_0 - \delta_1 < x < x_0$

$$H(x) > H(x_0) - \frac{\varepsilon}{4} = [H(x_0-0)+\varepsilon] - \frac{\varepsilon}{4} > [H(x) - \frac{\varepsilon}{4}] + \frac{3\varepsilon}{4} > H(x)$$

which is a contradiction. A similar argument is valid for $H(x_0-0) > H(x_0+0)$. □

<u>Theorem</u> 15. If $f(x,y,y')$ satisfies A_1, A_2, and A_3, then $H(x)$ is a solution of $y'' = f(x,y,y')$ on an open subset of [a,b] the complement of which is of measure zero.

<u>Proof</u>. Because $H(x)$ is a bounded subfunction, Corollary 5.1 gives that $H(x)$ has a finite derivative almost everywhere on [a,b]. Let $x_0 \varepsilon (a,b)$ be a point at which $H'(x_0)$ exists. This means there is a $\delta_0 > 0$ such that $[x_0 - \delta_0, x_0 + \delta_0]$

is contained in [a,b] and for all $0 < |x - x_0| \leq \delta_0$

$$\left|\frac{H(x) - H(x_0)}{x - x_0} - H'(x_0)\right| \leq 1$$

which shows that

$$\left|\frac{H(x) - H(x_0)}{x - x_0}\right| \leq |H'(x_0)| + 1.$$

Hence, for any $0 < \delta \leq \delta_0$

$$\begin{aligned}\left|\frac{H(x_0+\delta)-H(x_0-\delta)}{2\delta}\right| &\leq \left|\frac{H(x_0+\delta)-H(x_0)}{2\delta}\right| + \left|\frac{H(x_0)-H(x_0-\delta)}{2\delta}\right| \\ &\leq \frac{1}{2}[|H'(x_0)| + 1] + \frac{1}{2}[|H'(x_0)| + 1] \\ &= |H'(x_0)| + 1.\end{aligned}$$

Let $N = |H'(x_0)| + 1$. Since $H(x)$ is bounded on [a,b], set $M = \sup_{a \leq x \leq b} |H(x)|$. By Lemma 1 there is a $\delta(M,N) > 0$ for which the boundary value problem

$$y'' = f(x,y,y')$$
$$y(x_0-\delta) = H(x_0-\delta),\ y(x_0+\delta) = H(x_0+\delta)$$

has a solution of class $C^{(2)}$ on $[x_0-\delta, x_0+\delta]$ for any $0 < \delta \leq \delta(M,N)/2$. Let $\delta_1 = \min\{\delta_0, \delta(M,N)/2\}$ and $y_1(x)$ be the solution corresponding to $\delta = \delta_1$. Since $H(x)$ is simultaneously a subfunction and a superfunction on the interval $[x_0-\delta_1, x_0+\delta_1]$ and $y_1(x_0 \pm \delta_1) = H(x_0 \pm \delta_1)$, then $H(x) \leq y_1(x)$ and

$H(x) \geq y_1(x)$ on $(x_0-\delta_1, x_0+\delta_1)$ which shows that $H(x) \equiv y_1(x)$ on $[x_0-\delta_1, x_0+\delta_1]$. A closed interval such as this can be found about any point at which $H'(x)$ exists and can be extended to an open interval contained in $[a,b]$, at the endpoints of which $H'(x)$ does not exist finitely. Thus $H(x)$ is a solution on $[a,b]$ on an open set, the complement of which has measure zero by Corollary 5.1. $\square$

Now the right-hand and left-hand derivatives, $DH(x+0)$ and $DH(x-0)$, introduced in Corollary 5.1 are used to investigate some properties of $H(x)$ at points where the derivative does not exist.

Theorem 16. If $f(x,y,y')$ satisfies A_1, A_2, and A_3, then $DH(x+0) = DH(x-0)$ for all $x \varepsilon (a,b)$. Let E be the set of points in (a,b) at which $H(x)$ does not have a finite derivative. If $x \varepsilon E$ is a point of continuity of $H(x)$, either $DH(x+0) = DH(x-0) = +\infty$ or $DH(x+0) = DH(x-0) = -\infty$. If $H(x+0) > H(x-0)$, then $DH(x+0) = DH(x-0) = +\infty$; and if $H(x+0) < H(x-0)$, then $DH(x+0) = DH(x-0) = -\infty$.

Proof. If x is a point in (a,b) at which $H'(x)$ exists, then $DH(x+0) = DH(x-0) = H'(x)$. Now suppose $x_0 \varepsilon E$ and $H(x)$ is continuous at x_0. Assume that both $|DH(x_0+0)| < \infty$ and $|DH(x_0-0)| < \infty$. This means there are $\delta_+ > 0$ and $\delta_- > 0$ such that for any $0 < \delta < \delta_+$

$$\left| \frac{H(x_0+\delta) - H(x_0)}{\delta} - DH(x_0+0) \right| \leq 1$$

and for any $0 < \delta < \delta_-$

$$\left|\frac{H(x_0-\delta) - H(x_0)}{\delta} - DH(x_0-0)\right| \leq 1.$$

Then for $\delta \leq \min\{\delta_+, \delta_-\}$

$$\begin{aligned}\left|\frac{H(x_0+\delta)-H(x_0-\delta)}{2\delta}\right| &\leq \left|\frac{H(x_0+\delta)-H(x_0)}{2\delta}\right| + \left|\frac{H(x_0)-H(x_0-\delta)}{2\delta}\right| \\ &\leq \frac{1}{2}[|DH(x_0+0)| + 1] + \frac{1}{2}[|DH(x_0-0)| + 1] \\ &\equiv N\end{aligned}$$

and $N < +\infty$. Let $M = \sup_{a \leq x \leq b} |H(x)|$ and then Lemma 1 gives that there is a $\delta(M,N) > 0$ such that the boundary value problem

$$y'' = f(x,y,y')$$
$$y(x_0-\delta) = H(x_0-\delta),\ y(x_0+\delta) = H(x_0+\delta)$$

has a solution $y(x)$ of class $C^{(2)}$ on $[x_0-\delta, x_0+\delta]$ for all $0 < \delta \leq \min\{\delta_+, \delta_-, \delta(M,N)/2\}$. Using the same reasoning as in the proof of Theorem 15, $H(x)$ is of class $C^{(2)}[x_0-\delta, x_0+\delta]$ contradicting the fact that $x_0 \varepsilon E$. Hence, for any $x_0 \varepsilon E$ which is a point of continuity of $H(x)$, both $DH(x_0+0)$ and $DH(x_0-0)$ cannot be finite.

Suppose now $DH(x_0+0) = +\infty$, but $DH(x_0-0) < N_0 < +\infty$ for some $N_0 > 0$. Then there is a $\delta_0 > 0$ for which

$$\frac{H(x) - H(x_0)}{x - x_0} < N_0$$

whenever $x_0 - \delta_0 \leq x < x_0$. Since $x - x_0 < 0$,

$$H(x) - H(x_0) > N_0(x - x_0)$$

so that on $[x_0-\delta_0, x_0)$

$$H(x) > \omega_0(x) \equiv H(x_0) + N_0(x - x_0).$$

Since $DH(x_0+0) = +\infty$, there is a $\delta_1 > 0$ such that

$$\frac{H(x) - H(x_0)}{x - x_0} > N_0 + 2$$

for $x_0 < x \leq x_0+\delta_1$ or

$$H(x) > \omega_1(x) \equiv H(x_0) + (N_0 + 2)(x - x_0).$$

If $0 < \delta \leq \min\{\delta_0, \delta_1\}$,

$$\frac{\omega_1(x_0+\delta) - \omega_0(x_0-\delta)}{2\delta} = \frac{(N_0+2)\delta + N_0\delta}{2\delta} = N_0 + 1.$$

Let $M_1 = \max\{|\omega_1(x_0+\delta_1)|, |\omega_0(x_0-\delta_0)|, |H(x_0)|\}$ and $N_1 = N_0 + 1$. By Lemma 2 there is an $\eta(M_1, N_1)$ such that the boundary value problem

$$y'' = f(x,y,y')$$

$$y(x_0-\delta) = \omega_0(x_0-\delta),\ y(x_0+\delta) = \omega_1(x_0+\delta)$$

has a solution $y(x;\delta)$ of class $C^{(2)}$ on $[x_0-\delta, x_0+\delta]$ and

$|y'(x;\delta) - (N_0+1)| < 1$ for all $0 < \delta \leq \eta(M_1,N_1)/2$. Let $\delta_2 = \min\{\delta_0,\delta_1,\eta(M_1,N_1)/2\}$ and $y(x;\delta_2)$ be the solution corresponding to $\delta = \delta_2$. Suppose $y(x_0;\delta_2) \leq H(x_0)$. Then by the Mean Value Theorem there is an $x_0 < \bar{x} < x_0+\delta_2$ for which

$$\begin{aligned} y'(\bar{x};\delta_2) &= \frac{y(x_0+\delta_2,\delta_2) - y(x_0;\delta_2)}{\delta_2} \\ &\geq \frac{[H(x_0) + (N_0 + 2)\delta_2] - H(x_0)}{\delta_2} \\ &= N_0 + 2. \end{aligned}$$

However, because $|\bar{x} - x_0| < \eta(M_1,N_1)$, then

$$|y'(\bar{x};\delta_2)| \leq |y'(\bar{x};\delta_2) - (N_0 + 1)| + (N_0 + 1) < N_0 + 2$$

which is contradictory. This means $y(x_0;\delta_2) > H(x_0)$. Now since $\delta_2 \leq \min\{\delta_0,\delta_1\}$,

$$y(x_0-\delta_2;\delta_2) = \omega_0(x_0-\delta_2) < H(x_0-\delta_2)$$

and

$$y(x_0+\delta_2;\delta_2) = \omega_1(x_0+\delta_2) < H(x_0+\delta_2).$$

This implies that $y(x;\delta_2) \leq H(x)$ on $[x_0-\delta_2,x_0+\delta_2]$ because $H(x)$ is a superfunction. In particular, $y(x_0;\delta_2) \leq H(x_0)$ which is again contradictory. Thus $DH(x_0-0) = DH(x_0+0) = +\infty$. Similarly, if either $DH(x_0+0) = -\infty$ or $DH(x_0-0) = -\infty$, then $DH(x_0-0) = DH(x_0+0) = -\infty$.

Assume now that $x_0 \varepsilon E$ and $H(x_0+0) > H(x_0-0)$. By Corollary 14.1, $H(x_0) = H(x_0-0)$. Set $d = H(x_0+0) - H(x_0) > 0$. For some $0<m<+\infty$, suppose $DH(x_0+0) < m$. Then there is a $\delta_0^*>0$ such that

$$H(x) < \omega_0^*(x) \equiv H(x_0+0) + m(x-x_0)$$

for $x\varepsilon(x_0,x_0+\delta_0^*]$. If $\delta \leq \min\{\delta_0^*, d/m\}$, then

$$\frac{\omega_0^*(x_0+\delta) - [H(x_0+0) - m\delta]}{\delta} = \frac{2m\delta}{\delta} = 2m$$

and now let $M^* = \max\{|\omega_0^*(x_0+\delta_0^*)|, |H(x_0+0)-m\delta_0^*|\}$ and $N^* = 2m$. Hence, by Lemma 2 there is an $\eta(M^*,N^*) > 0$ such that the boundary value problem

$$y'' = f(x,y,y')$$
$$y(x_0) = H(x_0+0) - m\delta, \quad y(x_0+\delta) = \omega_0^*(x_0+\delta)$$

has a solution $y_1(x;\delta)$ for all $0 < \delta \leq \min\{\delta_0^*, d/m, \eta(M^*,N^*)\}$ and $|y_1'(x;\delta) - 2m| < 1$ for $x_0 \leq x \leq x_0 + \delta$. Let $\delta^* = \min\{\delta_0^*, d/m, \eta(M^*,N^*)\}$ and $y_1(x;\delta^*)$ be the corresponding solution. Since $H(x)$ is a subfunction, the relations

$$y_1(x_0;\delta^*) = H(x_0+0) - m\delta^* \geq H(x_0+0) - d = H(x_0)$$

and

$$y_1(x_0+\delta^*;\delta^*) = \omega_0^*(x_0+\delta^*) > H(x_0+\delta^*)$$

imply that $H(x) \le y_1(x;\delta^*)$ on $[x_o, x_o + \delta^*]$. Because $\delta^* \le \eta(M^*, N^*)$,

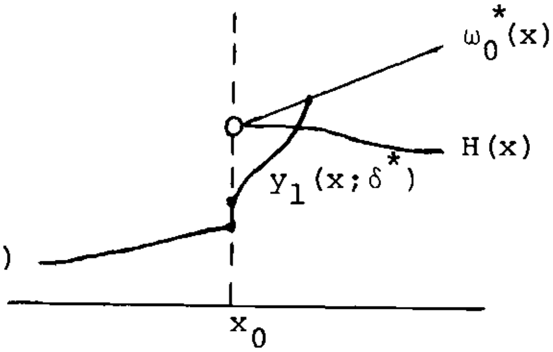
$$|y_1'(x_o;\ \delta^*)| \le |y_1'(x_o;\delta^*) - 2m| + 2m < 1 + 2m.$$

Hence, there is a $\delta_+^* > 0$ for which

$$\left| \frac{y_1(x_o+\delta;\delta^*) - y_1(x_o;\delta^*)}{\delta} \right| \le \left| y_1'(x_o;\delta^*) \right| + 1 \le 2m + 2$$

whenever $0 < \delta \le \min\ \{\delta_+^*,\ \delta^*\}$. Thus for such a value of δ

$$y_1(x_o + \delta;\ \delta^*) \le y_1(x_o;\ \delta^*) + (2m + 2)\delta.$$



By the definition of $H(x_o + 0)$ there is a $d^* > 0$ for which

$$H(x_o + \delta) - H(x_o + 0) > -\frac{1}{2}\, m\delta^*$$

for all $0 < \delta \le d^*$. Then for $0 < \delta \le \min\ \{d^*,\ \delta_+^*, \delta^*\}$,

$$\begin{aligned} H(x_o + \delta) &> H(x_o + 0) - \frac{1}{2}\, m\ \delta^* \\ &= [y_1(x_o;\ \delta^*) + m\ \delta^*] - \frac{1}{2}\, m\ \delta^* \\ &\ge y_1(x_o + \delta;\ \delta^*) - (2m + 2)\delta + \frac{1}{2}\, m\ \delta^*. \end{aligned}$$

If in addition $\delta < m\delta^*/2(2M + 2)$, this means $H(x_o+\delta) > y_1(x_o+\delta;\delta^*)$ which contradicts the fact that $H(x)$ is a subfunction. Thus $DH(x_o + 0) = +\infty$.

Assume that $DH(x_o - 0) < n < +\infty$ for some $n > 0$. This means that there is a $\delta_1^* > 0$ for which

$$H(x) > \omega_1^*(x) \equiv H(x_o) + n(x-x_o)$$

whenever $x_o - \delta_1^* \leq x < x_o$. Because $DH(x_o + 0) = +\infty$, there is a $\delta_2^* > 0$ for which

$$H(x) > H(x_o+0) + (n+2)(x-x_o) > \omega_2(x) \equiv H(x_o) + (n+2)(x-x_o)$$

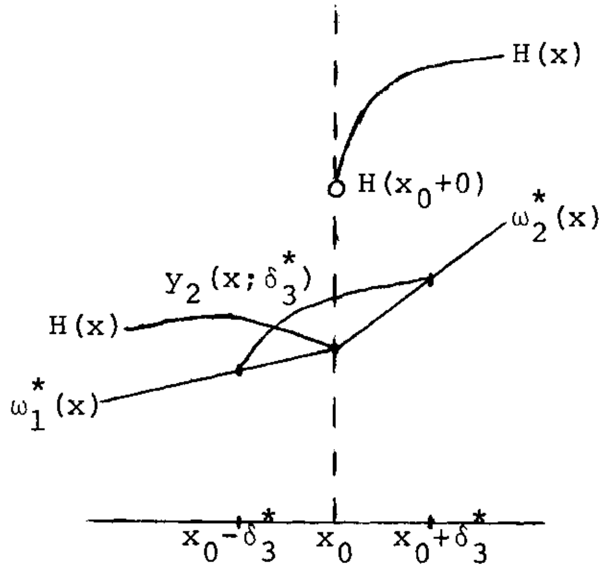
for $x_o < x \leq x_o + \delta_2^*$. If $\delta \leq \min\{\delta_1^*, \delta_2^*\}$,

$$\frac{\omega_2^*(x_o+\delta) - \omega_1^*(x_o-\delta)}{2\delta} = n + 1$$

and then for $M_1^* = \max\{|\omega_2^*(x_o+\delta_2^*)|, |\omega_1^*(x_o-\delta_1^*)|\}$ and $N_1^* = n+1$, Lemma 2 gives that there is an $\eta(M_1^*,N_1^*) > 0$ such that the boundary value problem

$$y'' = f(x,y,y')$$
$$y(x_o-\delta) = \omega_1^*(x_o-\delta),\ y(x_o+\delta) = \omega_2^*(x_o+\delta)$$

has a solution $y_2(x;\delta)$ of class $C^{(2)}$ and $|y_2'(x;\delta) - (n+1)| < 1$ on $[x_o-\delta, x_o+\delta]$ for all $0 < \delta \leq \eta(M_1^*, N_1^*)/2$. Set $\delta_3^* = \min\{\delta_1^*, \delta_2^*, \eta(M_1^*,N_1^*)/2\}$ and let $y_2(x;\delta_3^*)$ be the corresponding solution on $[x_o - \delta_3^*, x_o + \delta_3^*]$. By the Mean Value Theorem the same reasoning as for points of continuity of $H(x)$ gives that $y_2(x_o;\delta_3^*) > H(x_o)$.



However,

$$y_2(x_o-\delta_3^*;\ \delta_3^*) = \omega_1^*(x_o - \delta_3^*) < H(x_o-\delta_3^*)$$

and

$$y_2(x_o+\ \delta_3^*;\ \delta_3^*) = \omega_2^*(x_o+\delta_3^*) < H(x_o+\ \delta_3^*)$$

imply that $y(x_o;\ \delta_3^*) \le H(x_o)$ since $H(x)$ is a superfunction. This is contradictory. Therefore if $H(x_o+0) > H(x_o-0)$, then $DH(x_o+0) = DH(x_o-0) = +\infty$.

If $H(x_o+0) < H(x_o-0)$, then by analogous reasoning $DH(x_o+0) = DH(x_o=0) = -\infty$. $\square$

Attention is next shifted to the behavior of $H(x)$ at the endpoints of the compact interval $[a,b]$.

<u>Theorem</u> 17. Assume that $f(x,y,y')$ satisfies A_1, A_2, and A_3. If $DH(a+0) \neq +\infty$, then $H(a+0) = H(a)$. If $H(a+0) < \alpha$, $DH(a+0) = -\infty$. Thus if $DH(a+0)$ is finite,

$H(a+0) = H(a) = \alpha$. Similarly, if $DH(b=0) \neq +\infty$, then $H(b-0) = H(b)$. If $H(b-0) < \beta$, $DH(b-0) = -\infty$. Thus if $DH(b-0)$ is finite, $H(b-0) = H(b) = \beta$.

Proof. Suppose that $H(a) > H(a+0)$. Let $d=H(a) - H(a+0) > 0$. By the definition of $H(x)$, there is a continuous underfunction $\phi_1(x)$ on $[a,b]$ such that $H(a) - d/4 \leq \phi_1(a) \leq H(a)$. Hence, there is a $\delta_1 > 0$ for which $|\phi_1(x) - \phi_1(a)| < d/4$ whenever $a \leq x < a + \delta_1$. Similarly there is a $\delta_2 > 0$ such that $|H(x) - H(a+0)| < d/4$ for $a < x \leq a + \delta_2$. Setting $\delta_3 = \min\{\delta_1, \delta_2\}$ makes for $a < x \leq a + \delta_3$.

$$\begin{aligned} H(x) &\geq \phi_1(x) \\ &> \phi_1(a) - \frac{d}{4} \\ &\geq [H(a) - \frac{d}{4}] - \frac{d}{4} \\ &= [H(a+0) + d] - \frac{d}{2} \\ &> [H(x) - \frac{d}{4}] + \frac{d}{2} \\ &> H(x) \end{aligned}$$

which is impossible. Thus $H(a) \leq H(a+0)$.

Assume $DH(a+0) < m < \infty$ for some $m > 0$. Hence, there is a $\delta_o > 0$ such that for $a < x \leq a + \delta_o$

$$H(x) < \omega_o(x) \equiv H(a+0) + m(x-a).$$

Suppose $d_o = H(a+0) - H(a) > 0$. For any $0 < \delta \leq \delta_o$ with $m\delta \leq d_o$

$$\frac{\omega_o(a+\delta) - [H(a+0) - m\delta]}{\delta} = \frac{2m\delta}{\delta} = 2m.$$

Let $M = \max \{|H(a)|, |\omega_o(a+\delta_o)|\}$ and $N=2m$. Then by Lemma 1 there is a $\delta(M,N) > 0$ such that the boundary value problem

$$y'' = f(x,y,y')$$

$$y(a) = H(a+0) - m\delta, \quad y(a+\delta) = \omega_o(a+\delta)$$

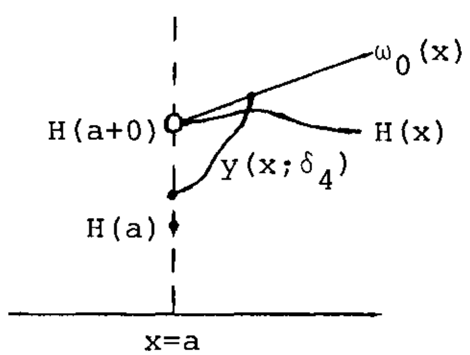
has a solution $y(x;\delta)$ of class $C^{(2)}$ on $[a,a+\delta]$ for all $0 < \delta \leq \delta(M,N)$. Set $\delta_4 = \min \{\delta_o, d_o/m, \delta(M,N)\}$ and denote $y(x;\delta_4)$ to be the corresponding solution for $\delta = \delta_4$.
Because $H(x)$ is a subfunction with

$$y(a;\delta_4) = H(a+0) - m\delta_4 \geq H(a+0) - d_o = H(a)$$

and

$$y(a+\delta_4;\delta_4) = \omega_o(a+\delta_4) > H(a+\delta_4),$$

then $H(x) \leq y(x;\delta_4)$ on $[a,a+\delta_4]$.



Now there is a $\delta_5 > 0$ for which $H(x) - H(a+0) > -m\delta_4/3$ whenever $a < x \leq a + \delta_5$ because $H(x)$ is a bounded function which implies $H(a+0)$ is a finite real number. By the continuity of $y(x;\ \delta_4)$ there is a $0 < \delta_6 \leq \delta_4$ such that $y(x;\delta_4) - y(a;\delta_4) < m\delta_4/3$ whenever $a < x \leq a+\delta_6$. Hence, for $|x-a| < \min\{\delta_5, \delta_6\}$ and $x > a$

$$H(x) > H(a+0) - \frac{m\delta_4}{3}$$

$$= [y(a;\delta_4) + m\delta_4] - \frac{m\delta_4}{3}$$

$$> [y(x;\delta_4) - \frac{m\delta_4}{3}] + \frac{2m\delta_4}{3}$$

$$> y(x;\delta_4)$$

which contradicts the fact that $H(x)$ is a subfunction. Thus $H(a+0) = H(a)$.

Now suppose $H(a+0) < \alpha$ and assume $DH(a+0) > n > -\infty$ for some $n > 0$. Then there is a $\delta_0^* > 0$ such that

$$H(x) > \omega(x) \equiv H(a+0) + n(x-a)$$

for $a < x \leq a + \delta_0^*$. Set $d_1 = \alpha - H(a+0)$. If $\delta \leq \min\{\delta_0^*, d_1/|n|\}$,

$$\frac{\omega(a+\delta) - [H(a+0) - n\delta]}{\delta} = \frac{2n\delta}{\delta} = 2n.$$

Let $M_1 = \max\{|H(a+0) - n\ \delta_0^*|, |\omega(a+\delta_0^*)|\}$ and $N_1 = 2|n|$. By Lemma 1 there is a $0 < \delta(M_1, N_1) \leq \delta_0^*$ for which the boundary value problem

$$y'' = f(x,y,y')$$

$$y(a) = H(a+0) - n\delta, \quad y(a+\delta) = \omega(a+\delta)$$

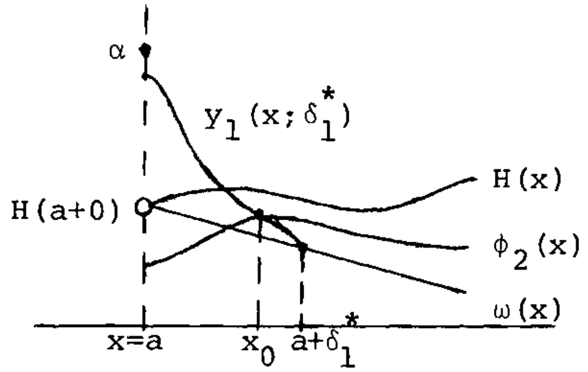
has a solution $y_1(x;\delta)$ of class $C^{(2)}$ on $[a,a+\delta]$ for all $0 < \delta \leq \delta(M_1,N_1)$. Let $\delta_1^* = \min\{\delta_0^*, d_1/|n|, \delta(M_1,N_1)\}$ and $y_1(x;\delta_1^*)$ be the corresponding solution. By the definition of $H(x)$ there is a continuous underfunction $\phi_2(x)$ on $[a,b]$ for which $\omega(a+\delta_1^*) < \phi_2(a+\delta_1^*) \leq H(a+\delta_1^*)$. Because

$$y_1(a;\delta_1^*) = H(a+0) - n\delta_1^* > H(a+0) \geq H(a) \geq \phi_2(a)$$

and

$$y_1(a+\delta_1^*;\ \delta_1^*) = \omega(a+\delta_1^*) < \phi_2(a+\delta_1^*)$$

and $y_1(x;\delta_1^*)$ and $\phi_2(x)$ are continuous on $[a,a+\delta_1^*]$, there is a point x_o in $(a,a+\delta_1^*)$ at which $y_1(x_o;\delta_1^*) = \phi_2(x_o)$ and $y_1(x;\delta_1^*) > \phi_2(x)$ on $[a,x_o)$.



Define the continuous function $\phi(x)$ on $[a,b]$ by

$$\phi(x) = \begin{cases} y_1(x;\delta_1^*) & \text{if } a \le x < x_o \\ \phi_2(x) & \text{if } x_o \le x \le b. \end{cases}$$

Suppose $y(x)$ is a solution of $y'' = f(x,y,y')$ on $[x_1,x_2] \subset [a,b]$ and $y(x_1) \ge \phi(x_1)$ and $y(x_2) \ge \phi(x_2)$. If $[x_1,x_2] \subset [a,x_o]$ then $y(x_i) \ge \phi(x_i) = y_1(x_i;\ \delta_1^*)$ for $i = 1,2$. Since $y_1(x;\delta_1^*)$ is a subfunction, this implies that $y(x) \ge y_1(x;\ \delta_1^*) = \phi(x)$ on $[x_1,x_2]$. Similarly if $[x_1,x_2] \subset [x_o,b]$, then $y(x_i) \ge \phi(x_i) = \phi_2(x_i)$ for $i = 1,2$ means that $y(x) \ge \phi_2(x) = \phi(x)$ on $[x_1,x_2]$ because $\phi_2(x)$ is a subfunction. Now let $a \le x_1 < x_o < x_2 \le b$. Since

$$y(x_1) \ge \phi(x_1) = y_1(x_1;\delta_1^*) > \phi_2(x_1)$$

and

$$y(x_2) \ge \phi(x_2) = \phi_2(x_2),$$

this implies $y(x) \ge \phi_2(x)$ on $[x_1,x_2]$. In particular, $y(x_0) \ge \phi_2(x_0) = y_1(x_0;\ \delta_1^*)$ which implies further that $y(x) \ge y_1(x;\delta_1^*)$ on $[x_1,x_0]$. Thus $y(x) \ge \phi(x)$ on $[x_1,x_2]$. These results mean that $\phi(x)$ is a continuous subfunction on $[a,b]$. Furthermore,

$$\phi(a) = y_1(a;\delta_1^*) = H(a+0) - n\delta_1^* \le H(a+0) - n\frac{d_1}{|n|} = \alpha$$

and $\phi(b) = \phi_2(b) \le \beta$ which imply that $\phi(x)$ is a continuous underfunction on $[a,b]$. However,

$$\phi(a) = y_1(a;\delta_1^*) = H(a+0) - n\delta_1^* > H(a+0) > H(a)$$

which contradicts the definition of H(x). Therefore $DH(a+0) = -\infty$.

Finally suppose DH(a+0) is finite. Then $H(a+0)=H(a)$ and $H(a+0) \geq \alpha$. By the definition of H(x), $H(a) \leq \alpha$. Thus $H(a+0) = H(a) = \alpha$. Analogous reasoning can be used to establish the results at x=b. □

Now that these properties of the function H(x) on [a,b] have been established, the concept of a generalized solution may be more clearly defined. A generalized solution to the second order boundary value problem

$$y'' = f(x,y,y')$$

$$y(a) = \alpha, \quad y(b) = \beta$$

is a function which has an absolutely continuous first derivative and which satisfies the differential equation almost everywhere on [a,b].

Suppose y(x) is a solution of this boundary value problem of class $C^{(2)}$ on (a,b) and $C^{(1)}$ on [a,b]. Since $H(a) \leq \alpha = y(a)$ and $H(b) \leq \beta = y(b)$, then $H(x) \leq y(x)$ on [a,b] by the subfunction nature of H(x). Furthermore, y(x) is also a continuous underfunction which means by the definition of H(x) that $H(x) \geq y(x)$ on [a,b]. Thus if f(x,y,y') satisfies A_1, A_2 and A_3 and if the boundary value problem has a solution of class $C^{(2)}(a,b) \cap C^{(1)}[a,b]$, then the generalized solution is this solution.

CHAPTER V

SOME EXISTENCE AND UNIQUENESS RESULTS

Now that the concept of a generalized solution H(x) of the boundary value problem

$$y'' = f(x,y,y') \qquad y(a) = \alpha, \quad y(b) = \beta \tag{5-1}$$

has been presented and certain properties of it have been established, conditions on f(x,y,y') are investigated that make H(x) a solution of this boundary value problem.

To begin this investigation consider the special case of the differential equation $y'' = g(x,y)$.

<u>Theorem</u> 18. Assume that g(x,y) satisfies A_1 on [a,b]. Then the generalized solution H(x) is bounded on [a,b] and is simultaneously a subfunction and a superfunction with respect to solutions of

$$y'' = g(x,y) \qquad y(a) = \alpha, \; y(b) = \beta. \tag{5-2}$$

Furthermore, H(x) is of class $C^{(2)}[a,b]$ and is the unique solution of the boundary value problem on [a,b].

<u>Proof</u>. By Theorem 12 there is an underfunction

$\phi_o(x)$ of class $C^{(2)}[a,b]$ satisfying $\underline{D}\phi_o'(x) > g(x,\phi_o(x))$ on $[a,b]$ with $\phi_o(a) = \alpha$ and $\phi_o(b) = \beta$ and an overfunction $\Psi_o(x)$ of class $C^{(2)}[a,b]$ satisfying $\overline{D}\Psi_o'(x) < g(x,\Phi_o(x))$ on $[a,b]$ with $\Psi_o(a) = \alpha$ and $\Psi_o(b) = \beta$ with respect to (5-2). Then $H(x)$ exists and by Theorem 8 $\phi_o(x) \leq H(x) \leq \Psi_o(x)$ on $[a,b]$ since $\overline{D}\Psi_o'(x) < g(x,\Psi_o(x))$ makes $\phi(x) \leq \Psi_o(x)$ for every continuous underfunction $\phi(x)$ on $[a,b]$. Thus $H(x)$ is bounded on $[a,b]$. Theorem 3 guarantees that $H(x)$ is a subfunction on $[a,b]$. Again using the fact that $\overline{D}\Psi_o'(x) < g(x,\Psi_o(x))$ and $g(x,y)$ satisfies A_1, $H(x)$ can be shown to be a superfunction also on $[a,b]$ by a proof analogous to that of Theorem 14.

Since $H(x)$ is bounded, there is an $M > 0$ such that $|H(x)| \leq M$ for all $x \in [a,b]$. By Lemma 3 there is a $\delta(M) > 0$ for which the boundary value problem

$$y'' = g(x,y)$$
$$y(x_1) = y_1,\ y(x_2) = y_2$$

has a solution of class $C^{(2)}$ on $[x_1,x_2] \subset [a,b]$ whenever $|x_2-x_1| \leq \delta(M)$, $|y_1| \leq M$, and $|y_2| \leq M$. Hence, a sequence of points $x_o = a < x_1 < \ldots < x_{n-1} < x_n = b$ can be chosen with the property that $|x_k-x_{k-1}| \leq \delta(M)$ for $k = 1,2,\ldots,n$. Denote by $y_k(x)$ the solution of the boundary value problem

$$y'' = g(x,y)$$

$$y(x_{k-1}) = H(x_{k-1}),\ y(x_k) = H(x_k)$$

for $k = 1,2,\ldots,n$. Because $H(x)$ is simultaneously a subfunction and a superfunction on $[a,b]$, then $H(x) \leq y_k(x)$ and $H(x) \geq y_k(x)$ for $x_{k-1} \leq x \leq x_k$ which means that $H(x) = y_k(x)$ on $[x_{k-1}, x_k]$ for $k = 1,2,\ldots,n$. Let $x_j \varepsilon \{x_1,x_2,\ldots,x_{n-1}\}$ and choose points $x^*_{j-1} \varepsilon (x_{j-1}, x_j)$ and $x^*_{j+1} \varepsilon (x_j,x_{j+1})$ for which $|x^*_{j+1} - x^*_{j-1}| \leq \delta(M)$. Then by Lemma 3 there is a solution $\overline{y}_j(x)$ of the boundary value problem

$$y'' = g(x,y)$$
$$y(x^*_{j-1}) = H(x^*_{j-1}),\ y(x^*_{j+1}) = H(x^*_{j+1})$$

of class $C^{(2)}$ $[x^*_{j-1}, x^*_{j+1}]$. By the subfunction and superfunction nature of $H(x)$, $H(x) = \overline{y}_j(x)$ on $[x^*_{j-1}, x^*_{j+1}]$. Let

$$y_{j,j+1}(x) = \begin{cases} y_j(x) & \text{if } x_{j-1} \leq x \leq x_j \\ y_{j+1}(x) & \text{if } x_j \leq x \leq x_{j+1} . \end{cases}$$

Hence, $y_{j,j+1}(x) = H(x) = \overline{y}_j(x)$ on $x_{j-1} < x^*_{j-1} \leq x \leq x^*_{j+1} < x_{j+1}$ which implies that $y_{j,j+1}(x) = H(x)$ is of class $C^{(2)}[x_{j-1},x_{j+1}]$. Thus $H(x) = y_k(x)$ for $k = 1,2,\ldots,n$ is a solution of $y'' = g(x\ ,y)$ of class $C^{(2)}$ $[a,b]$. Furthermore, since $\phi_o(x) \leq H(x) \leq \Psi_o(x)$ on $[a,b]$ and $\phi_o(a) = \Psi_o(a) = \alpha$ and $\phi_o(b) = \Psi_o(b) = \beta$, then $H(a) = \alpha$ and $H(b) = \beta$ which shows that $H(x)$ is a solution of the boundary value problem (5-2). If $y_o(x)$ is another solution of class $C^{(2)}[a,b]$ of (5-2), the fact that $H(x)$ is simultaneously a subfunction and a superfunction again shows that $H(x) = y_o(x)$. Therefore $H(x)$ is the unique solution of class $C^{(2)}[a,b]$. $\square$

All of the remaining results refer to the more general boundary value problem (5-1). The next existence result is for solutions of $y'' = f(x,y,y')$ on (a,b).

<u>Theorem</u> 19. If $f(x,y,y')$ satisfies A_1, A_2, and A_3, then the generalized solution $H(x)$ has a finite derivative at each point of (a,b).

<u>Proof</u>. Let $x_0 \varepsilon (a,b)$. To investigate the nature of $H'(x_0)$, the cases $H(x_0) > 0$ and $H(x_0) \leq 0$ are considered separately.

Suppose first that $H(x_0)>0$. Then $H(x_0)=\min\{H(x_0-0),H(x_0+0)\}$ by Corollary 14.1. Now assume $H(x_0) = H(x_0-0)$ which does not give any loss of generality. If $y(x)$ is any solution of $y'' = -(K + \rho)y'$ such that $y(x) \geq 0$ and $y'(x) \geq 1$ where $\rho = \max_{a \leq x \leq b} |f(x,0,0)|$, then by A_3

$$
\begin{aligned}
\overline{D}y'(x) &= - (K + \rho)y'(x) \\
&= - Ky'(x) - \rho y'(x) \\
&\leq f(x,0,y'(x)) - f(x,0,0) - \rho y'(x).
\end{aligned}
$$

Furthermore, since $-f(x,0,0) \leq \rho$ and since $y(x) \geq 0$ and $y'(x) \geq 1$,

$$
\begin{aligned}
\overline{D}y'(x) &\leq f(x,0,y'(x)) + \rho - \rho y'(x) \\
&\leq f(x,0,y'(x)) \\
&\leq f(x,y(x),y'(x)).
\end{aligned}
$$

Thus y(x) is a superfunction by Theorem 7'.

If

$$y''(x) = -(K+\rho)y'(x),$$

then

$$y'(x) = Ce^{-(K+\rho)x}$$

where C is an integration constant. In order that $y'(x) \geq 1$ on $[x_0,b]$, set $C = e^{(K+\rho)b}$ which makes

$$y'(x) = e^{-(K+\rho)(x-b)}.$$

Let $x\varepsilon[x_0,b]$. Integration on $[x_0,x]$ gives

$$y(x)-y(x_0)=\int_{x_0}^{x} e^{-(K+\rho)(t-b)}dt=-\frac{1}{K+\rho}[e^{-(K+\rho)(x-b)}-e^{-(K+\rho)(x_0-b)}].$$

Set $y(x_0) = H(x_0+0)$ and denote

$$y_1(x) = H(x_0+0) - \frac{1}{K+\rho}[e^{-(K+\rho)(x-b)} - e^{-(K+\rho)(x_0-b)}].$$

Similarly integration on [x,b] gives

$$y(b) - y(x) = \int_{x}^{b} e^{-(K+\rho)(t-b)}dt = -\frac{1}{K+\rho}[1 - e^{-(K+\rho)(x-b)}].$$

Set y(b) = B and denote

$$y_2(x) = B + \frac{1}{K+\rho}[1 - e^{-(K+\rho)(x-b)}].$$

Hence, any solution of $y'' = -(K+\rho)y'$ on $[x_0,b]$ has the form $c_1y_1(x) + c_2y_2(x)$. Let $y_+(x)$ be the solution with the properties $y_+(x_0) = H(x_0+0)$, $y_+(b) = B$, and $y_+'(x) \geq 1$. To satisfy the boundary conditions for $y_+(x)$, set

$$H(x_0+0) = c_1y_1(x_0) + c_2y_2(x_0)$$

$$= c_1H(x_0+0) + c_2\{B + \frac{1}{K+\rho}[1 - e^{-(K+\rho)(x_0-b)}]\}$$

and

$$B = c_1y_1(b) + c_2y_2(b)$$

$$= c_1\{H(x_0+0) - \frac{1}{K+\rho}[1 - e^{-(K+\rho)(x_0-b)}]\} + c_2B$$

which have the solution

$$c_1 = \frac{-B(K+\rho)}{1 - e^{-(K+\rho)(x_0-b)}}, \qquad c_2 = \frac{H(x_0+0)(K+\rho)}{1 - e^{-(K+\rho)(x_0-b)}}$$

Then

$$y_+(x) = \frac{K+\rho}{1-e^{-(K+\rho)(x_0-b)}}\{-BH(x_0+0) + \frac{B}{K+\rho}[e^{-(K+\rho)(x-b)} - e^{-(K+\rho)(x_0-b)}]$$

$$+H(x_0+0)B + \frac{H(x_0+0)}{K+\rho}[1-e^{-(K+\rho)(x-b)}]\}$$

$$= \frac{1}{1-e^{-(K+\rho)(x_0-b)}}\{B[e^{-(K+\rho)(x-b)} - e^{-(K+\rho)(x_0-b)}]$$

$$+H(x_0+0)[1-e^{-(K+\rho)(x-b)}]\}.$$

Since

$$y_+'(x) = \frac{(K+\rho)e^{-(K+\rho)(x-b)}}{1 - e^{-(K+\rho)(x_0-b)}}[-B + H(x_0+0)]$$

$$= \frac{(K+\rho)e^{-(K+\rho)(x-b)}}{e^{-(K+\rho)(x_0-b)} - 1}[B - H(x_0+0)]$$

and $e^{-(K+\rho)(x_0-b)} - 1 > 0$, then to satisfy the condition $y_+'(x) \geq 1$ on $[x_0,b]$ requires

$$B \geq H(x_0+0) + \frac{e^{-(K+\rho)(x_0-b)} - 1}{K + \rho} .$$

Because $H(x_0+0) \geq H(x_0) > 0$, then $B > 0$ which makes $y_+(x) \geq 0$ on $[x_0,b]$. Suppose also that $B \geq H(b)$, and now set

$$B = \max \{H(b), H(x_0+0) + \frac{e^{-(K+\rho)(x_0-b)} - 1}{K + \rho}\}.$$

Assume that for some x_1 in (x_0,b) that $H(x_1)-y_+(x_1)=\varepsilon>0$. By the definition of $H(x)$, there is a continuous underfunction $\phi(x)$ for which $H(x_1)-\phi(x_1)\leq\varepsilon/2$. Since $\phi(b)\leq H(b)\leq y_+(b)$ and $\phi(x_0)\leq H(x_0)\leq H(x_0+0)=y_+(x_0)$, Theorem 8 implies that $\phi(x) \leq y_+(x)$ on $[x_0,b]$. However, at x_1

$$\phi(x_1) \geq H(x_1) - \frac{\varepsilon}{2} = [y_+(x_1) + \varepsilon] - \frac{\varepsilon}{2} > y_+(x_1)$$

which is a contradiction. Thus $H(x) \leq y_+(x)$ on $[x_0,b]$.

Then for $x_0 < x \leq b$

$$\frac{H(x) - H(x_0+0)}{x - x_0} = \frac{H(x) - y_+(x_0)}{x - x_0} \leq \frac{y_+(x) - y_+(x_0)}{x - x_0},$$

and so

$$DH(x_0+0) = \lim_{x \to x_0+} \frac{H(x) - H(x_0+0)}{x - x_0} \leq y_+'(x_0) < +\infty.$$

Hence, $H(x_0+0) \leq H(x_0-0)$ by Theorem 16.

By a very similar proof use of the superfunction defined on $[a,x_0]$

$$y_-(x) = \frac{1}{1 - e^{(K+\rho)(x_0-a)}}\{A[e^{(K+\rho)(x-a)} - e^{(K+\rho)(x_0-a)}]$$

$$+ H(x_0)[1 - e^{(K+\rho)(x-a)}]\}$$

where

$$A = \max\{H(a), H(x_0) + \frac{e^{(K+\rho)(x_0-a)} - 1}{K + \rho}\}$$

shows that $DH(x-0) > -\infty$. This makes $H(x_0+0) \geq H(x_0-0)$ by Theorem 16. Thus $H(x)$ is continuous at x_0 and by Theorem 16 again $H'(x_0)$ exists and is finite.

The case of $H(x_0) \leq 0$ is analogously proved using first the subfunction defined on $[a,x_0]$

$$y_-^*(x) = \frac{1}{e^{(K+\rho)(x_0-a)} - 1}\{A^*[e^{(K+\rho)(x_0-a)} - e^{(K+\rho)(x-a)}]$$

$$+ H(x_0)[e^{(K+\rho)(x-a)} - 1]\}$$

where

$$A^* = \min\ \{H(a), H(x_0) - \frac{e^{(K+\rho)(x_0-a)} - 1}{K + \rho}\}$$

and then the subfunction defined on $[x_0,b]$

$$y_+^*(x) = \frac{1}{e^{-(K+\rho)(x_0-b)} - 1}\{B^*[e^{-(K+\rho)(x_0-b)} - e^{-(K+\rho)(x-b)}]$$

$$+ H(x_0)[e^{-(K+\rho)(x-b)} - 1]\}$$

where

$$B^* = \min\ \{H(b), H(x_0) - \frac{e^{-(K+\rho)(x_0-b)} - 1}{K + \rho}\}.$$

Therefore $H'(x)$ exists and is finite on (a,b). $\square$

<u>Corollary</u> 19.1. If $f(x,y,y')$ satisfies A_1, A_2, and A_3, then $H(x)$ is of class $C^{(2)}$ and is a solution of $y''=f(x,y,y')$ on (a,b).

<u>Proof</u>. By Theorem 19 $H(x)$ is continuous and $H'(x)$ exists with finite values on (a,b). Let $x_0 \varepsilon (a,b)$. Theorem 15 gives that there is an open interval (x_1,x_2) of positive length such that $a \leq x_1 < x_0 < x_2 \leq b$ and $H(x)$ is a solution of $y'' = f(x,y,y')$ of class $C^{(2)}$ on (x_1,x_2). Suppose that this interval cannot be extended to (a,b). For definiteness,

assume there is a first point $x_3 \varepsilon (a,x_1)$ at which $H(x)$ is no longer a solution. Now $H'(x_3)$ is finite which means that there is an open interval (t_1,t_2) of positive length containing x_3 and $H(x)$ is of class $C^{(2)}$ on (t_1,t_2). Since (x_3,x_2) and (t_1,t_2) overlap, $H(x)$ is of class $C^{(2)}$ at x_3 which means (x_1,x_2) can be extended to at least (t_1,x_2). This contradicts the choice of x_3. Thus $H(x)$ is a solution of class $C^{(2)}$ on (a,b). $\square$

Further existence results can be found after establishing a connection with an initial condition.

<u>Theorem</u> 20. Let T be any compact subset of $R_2 = \{(x,y) \mid a \leq x \leq b, |y| < \infty\}$ and let $\nu(t)$ be a positive continuous function of t for $t \geq 0$ such that

$$|f(x,y,y') - f(x,y,0)| \leq K_T \nu(|y'|) \tag{5-3}$$

and

$$\int_0^\infty \frac{t}{\nu(t)+1} dt = +\infty$$

where K_T is a constant depending on T for $(x,y)\varepsilon T$ and $|y'| < \infty$. Then for any $A > 0$ there exists a $B_{A,T} > 0$ such that $|y'(x)| < B_{A,T}$ for any solution $y(x)$ of $y'' = f(x,y,y')$ with the initial conditions $y(x_0) = y_0$ and $|y'(x_0)| \leq A$ for $(x_0,y_0)\varepsilon T$ as long as $(x,y(x))\varepsilon T$.

<u>Proof</u>. Let T be any compact subset of R_2. Then for any (x_1,y_1), $(x_2,y_2)\varepsilon T$, there is an $L > 0$ such that $|y_2-y_1| < L$.

Hence, for any $A > 0$ there is a $B_{A,T} > 0$ for which

$$\int_A^{B_{A,T}} \frac{t}{\nu(t)+1} dt > LK$$

where $K = \max \{K_T, \max_{(x,y)\varepsilon T} |f(x,y,0)|\}$.

Let $y(x)$ be an arbitrary solution of $y'' = f(x,y,y')$ with initial conditions $y(x_0) = y_0$, $y'(x_0) = m$ where (x_0,y_0) is in T and $0 < m \leq A$. Suppose that $x_0 \varepsilon [x_1,x_2]$ and $(x,y(x)) \varepsilon T$ for all $x \varepsilon [x_1,x_2]$. Assume now that $y'(x) \geq B_{A,T}$ for some $x \varepsilon [x_1,x_2]$. Since $y(x)$ is a solution, both $y(x)$ and $y'(x)$ are continuous. This means there are two points t_1 and t_2 in $[x_1,x_2]$ such that $y'(t_1) = A$, $y'(t_2) = B_{A,T}$, and $A \leq y'(x) \leq B_{A,T}$ for $t_1 \leq x \leq t_2$. With no loss in generality, assume $t_1 < t_2$ which makes $y''(x) \geq 0$ on $[t_1,t_2]$.

Since $(x,y(x)) \varepsilon T$ and $|y'(x)| < \infty$ for $t_1 \leq x \leq t_2$, then (5-3) implies that on $[t_1,t_2]$

$$\begin{aligned} y''(x) &= f(x,y(x),y'(x)) \\ &\leq K_T \nu[y'(x)] + f(x,y(x),0) \\ &\leq K \nu [y'(x)] + K \\ &= K(\nu[y'(x)] + 1). \end{aligned}$$

Hence,

$$y'(x)y''(x) \leq Ky'(x)(\nu[y'(x)] + 1)$$

or

$$\frac{y'(x)y''(x)}{\nu[y'(x)] + 1} \leq Ky'(x).$$

Integration then gives that

$$\int_{t_1}^{t_2} \frac{y'(x)y''(x)}{\nu[y'(x)] + 1}dx \leq \int_{t_1}^{t_2} Ky'(x)ds = K[y(t_2) - y(t_1)] < KL$$

because $0<A\leq y'(x)$ on $[t_1,t_2]$. However, by the choice of A and $B_{A,T}$

$$\int_{t_1}^{t_2} \frac{y'(x)y''(x)}{\nu[y'(x)] + 1}dx = \int_{A}^{B_{A,T}} \frac{s}{\nu(s) + 1}ds > LK$$

which is contradictory. Thus $y'(x) < B_{A,T}$ on $[x_1,x_2]$.

Analogously it can be shown that $y'(x) > -B_{A,T}$ on $[x_1,x_2]$ whenever $y'(x_0) \geq -A$. $\square$

These two results for the differential equation $y'' = f(x,y,y')$ can now be combined into an existence and uniqueness theorem for (5-1).

Theorem 21. Let $f(x,y,y')$ satisfy A_1, A_2, and A_3; and suppose there is a positive continuous function $\nu(t)$ for $t\geq 0$ such that

$$|f(x,y,y') - f(x,y,0)| \leq K_T\nu(|y'|)$$

and

$$\int_0^{\infty} \frac{t}{\nu(t)+1}dt = +\infty$$

for K_T a constant depending on compact subsets T of R_2 where $(x,y)\varepsilon T$ and $|y'| < \infty$. Then the generalized solution H(x) is the unique solution of class $C^{(2)}[a,b]$ of the boundary value problem

$$y'' = f(x,y,y')$$
$$y(a) = \alpha, \quad y(b) = \beta.$$

Proof. By Theorem 13 $H(x)$ is a bounded subfunction on $[a,b]$, and let $M > 0$ be chosen so that $|H(x)| \leq M$ on $[a,b]$. Set $T = \{(x,y) | \ a \leq x \leq b, \ |y| \leq M$. Theorem 19 ensures the existence of a point x_0 in (a,b) at which $H'(x_0)$ exists as a finite value and $H'(x)$ exists finitely on (a,b). Hence, by Theorem 20 there exists a $B_{|H'(x_0)|,T} > 0$ for which $|H'(x)| < B_{|H'(x_0)|,T}$ for all $a \leq x \leq b$. Theorems 16 and 17 imply that $H(x)$ is continuous on $[a,b]$ and satisfies the boundary conditions. Using this bound on $|H'(x)|$ and the boundedness of $|H(x)|$ on $[a,b]$, then Lemma 1 allows a method analogous to that used in the proof of Theorem 18 to show that $H(x)$ is a solution of class $C^{(2)}[a,b]$ of (5-1). Corollary 7.2 guarantees that $H(x)$ is the unique solution. $\square$

Corollary 21.1. Suppose that $f(x,y,y')$ satisfies A_1 and

$$|f(x,y,y_1') - f(x,y,y_2')| \leq K_T|y_1' - y_2'|$$

where K_T is a constant depending on compact subsets T of R_2 and $(x,y) \in T$ and $|y_1'|, |y_2'| < \infty$. Then $H(x)$ is the solution of the boundary value problem of class $C^{(2)}$ $[a,b]$.

Proof. Let Q_3 be any compact subset of R_3 and let

(x,y,y_1'), (x,y,y_2') be points in Q_3. Define the compact set T_{Q_3} by

$$T_{Q_3} = \{(x,y) \mid (x,y,y') \varepsilon Q_3\} \subset R_2.$$

Then there is a $K_{T,Q_3} > 0$ for which

$$|f(x,y,y_1') - f(x,y,y_2')| \leq K_{T,Q_3} |y_1' - y_2'|$$

which implies that $f(x,y,y')$ satisfies A_2. Furthermore, define the compact set $T_{[a,b]} = \{(x,0) | \leq a \leq x \leq b\} \subset R_2$ which means there is a $K_{[a,b]} > 0$ such that for $x \varepsilon [a,b]$ and $|y'| < \infty$

$$|f(x,0,y') - f(x,0,0)| \leq K_{[a,b]} |y'|.$$

Hence, $f(x,y,y')$ also satisfies A_3. Finally let T be any compact subset of R_2. Then there exists a $K_T > 0$ for which

$$|f(x,y,y') - f(x,y,0)| \leq K_T |y'|$$

for $(x,y) \varepsilon T$ and $|y'| < \infty$. Setting $\nu(t) = t$ makes this condition have the form of (5-3). For $A > 0$ integration over $[0,A]$ results in

$$\int_0^A \frac{t}{\nu(t)+1} dt = \int_0^A \frac{t}{t+1} dt$$

$$= \int_0^A (1 - \frac{1}{t+1}) dt$$

$$= A - \ln (A+1)$$

$$= \ell n\ e^A - \ell n(A+1)$$

$$= \ell n\ \frac{e^A}{A+1} .$$

Hence,

$$\int_0^\infty \frac{t}{\nu(t)+1}\, dt = \lim_{A\to\infty} \ell n\ \frac{e^A}{A+1} = +\ \infty.$$

Then Theorem 21 shows that $H(x)$ is the solution of the boundary value problem (5-1). $\square$

<u>Corollary</u> 21.2. Suppose that $f(x,y,y')$ satisfies A_1 and for all (x,y,y_1'), (x,y,y_2') in R_3 there is a $K > 0$ such that

$$|f(x,y,y_1') - f(x,y,y_2')| \le K\ |y_1' - y_2| .$$

Then $H(x)$ is the solution of the boundary value problem of class $C^{(2)}[a,b]$.

<u>Proof</u>. Let T be a compact subset of R_2 and let $|y_1'|$, $|y_2'| < \infty$. Then (x,y,y_1'), (x,y,y_2') are in R_3 for any $(x,y,)\varepsilon T$. Hence, for $K_T = K$

$$|f(x,y,y_1') - f(x,y,y_2')| \le K_T\ |y_1' - y_2'| .$$

Thus by Corollary 21.1 $H(x)$ is the solution of class $C^{(2)}[a,b]$ of (5-1). $\square$

As an example of these results, consider certain central force problems from classical mechanics where frictional forces of the form $b\ r'(t)$ are present. The equation of motion for this situation is

$$\mu r''(t) = F[r(t)] + br'(t)$$

where b and μ are constants. If $F[r(t)]$ is an increasing function in terms of $r(t)$, then this second order differential equation has a unique solution for each set of finite boundary conditions $r(t_1) = r_1$ and $r(t_2) = r_2$. In particular, using the force of gravitational attraction between bodies of mass m and M

$$F[r(t)] = -\frac{GmM}{[r(t)]^2},$$

this equation of motion becomes

$$r'' = -\frac{GmM}{\mu[r(t)]^2} + \frac{b}{\mu} r'(t)$$

where G is the gravitational constant and μ is the reduced mass defined by

$$\frac{1}{\mu} = \frac{1}{m} + \frac{1}{M} = \frac{M+m}{mM}.$$

Thus the boundary value problem

$$r''(t) = -\frac{G(M+m)}{[r(t)]^2} + \frac{b(M+m)}{mM} r'(t)$$

$$r(t_1) = r_1,\ r(t_2) = r_2$$

has a unique solution for any choice of t_1, t_2, r_1, and r_2 by Corollary 21.2 using $K = b(M+m)/mM$.

For the case of zero boundary conditions, $y(a)=y(b)=0$, the requirements on $f(x,y,y')$ can be slightly reduced and still ensure a unique solution of this boundary value problem as this final result shows.

Theorem 22. If $f(x,y,y')$ satisfies A_1, A_2, and A_3, then the generalized solution $H(x)$ is the solution of the boundary value problem

$$y'' = f(x,y,y')$$
$$y(a) = 0,\ y(b) = 0.$$

Proof. By Corollary 19.1 $H(x)$ is of class $C^{(2)}$ and a solution of $y'' = f(x,y,y')$ on (a,b). To complete this proof a continuous underfunction $\phi(x)$ and a continuous overfunction $\Psi(x)$ satisfying the zero boundary conditions are constructed.

Consider the differential equation $y'' = (K+\rho)y'$ on $[a,b]$ where $\rho = \max\limits_{a\leq x\leq b} |f(x,0,0)|$. Then $y'(x) = e^{(K+\rho)(x-a)}$ is a solution with the property that $y'(x) \geq 1$. Denote by $\phi_1(x)$ the solution for which $\phi_1(b) = 0$, and then integration over $[x,b]$ for $a \leq x \leq b$ gives

$$\phi_1(b) - \phi_1(x) = \frac{1}{K+\rho}\ [e^{(K+\rho)(b-a)} - e^{(K+\rho)(x-a)}]$$

or

$$\phi_1(x) = \frac{1}{K+\rho}\ [e^{(K+\rho)(x-a)} - e^{(K+\rho)(b-a)}]$$

It follows that $\phi_1(x) \leq 0$ and $\phi_1'(x) \geq 1$ on [a,b]. Since

$$\begin{aligned} \underline{D}\phi_1'(x) &= (K + \rho)\ \phi_1'(x) \\ &= K\phi_1'(x) + \rho\phi_1'(x) \\ &\geq f(x,0,\phi_1'(x)) - f(x,0,0) + \rho\phi_1'(x) \\ &\geq f(x,0,\phi_1'(x)) - f(x,0,0) + \rho \\ &\geq f(x,0,\phi_1'(x)) \\ &\geq f(x,\phi_1(x),\ \phi_1'(x)), \end{aligned}$$

$\phi_1(x)$ is a continuous subfunction with respect to $y'' = f(x,y,y')$ on [a,b] by Theorem 7. In a like manner, the solution $\phi_2(x)$ of $y'' = -(K+\rho)y'$ given by

$$\phi_2(x) = \frac{1}{K+\rho}\ [e^{(K+\rho)(b-x)} - e^{(K+\rho)(b-a)}]$$

has the properties $\phi_2(x) \leq 0$ and $\phi_2'(x) \leq -1$ on [a,b] and $\phi_2(a) = 0$. Since

$$\begin{aligned} \underline{D}\phi_2'(x) &= -\ (K+\rho)\ \phi_2'(x) \\ &= -\ K\phi_2'(x) - \rho\phi_2'(x) \\ &\geq f(x,0,\phi_2'(x)) - f(x,0,0) + \rho \\ &\geq f(x,\phi_2(x),\ \phi_2'(x)), \end{aligned}$$

$\phi_2(x)$ is also a continuous subfunction on [a,b]. Hence by Theorem 3

$$\phi(x) = \max \{\phi_1(x), \phi_2(x)\}$$

is a continuous subfunction on [a,b]. At the endpoints

$$\phi(a) = \max \{\frac{1}{K+\rho} [1 - e^{(K+\rho)(b-a)}], 0\} = 0$$

and

$$\phi(b) = \max \{0, \frac{1}{K+\rho} [1 - e^{(K+\rho)(b-a)}]\} = 0.$$

Similarly the solution $\Psi_1(x)$ of $y'' = (K+\rho)y'$ for which $\Psi_1(x) \geq 0$ and $\Psi_1'(x) \leq -1$ on [a,b] with $\Psi_1(b) = 0$ given by

$$\Psi_1(x) = \frac{1}{K+\rho} [e^{(K+\rho)(b-a)} - e^{(K+\rho)(x-a)}]$$

and the solution $\Psi_2(x)$ of $y'' = -(K+\rho)y'$ for which $\Psi_2(x) \geq 0$ and $\Psi_2(x) \geq 1$ on [a,b] with $\Psi_2(a) = 0$ given by

$$\Psi_2(x) = \frac{1}{K+\rho} [e^{(K+\rho)(b-a)} - e^{(K+\rho)(b-x)}]$$

are continuous superfunctions on [a,b]. Then

$$\Psi(x) = \min \{\Psi_1(x), \Psi_2(x)\}$$

is a continuous superfunction on [a,b] with $\Psi(a) = \Psi(b) = 0$.

Using this continuous underfunction $\phi(x)$ and continuous overfunction $\psi(x)$, the definition of $H(x)$ and Theorem 8 imply that $\phi(x) \leq H(x) \leq \Psi(x)$ on [a,b]. Hence, $H(a) = 0$ and $H(b) = 0$. Also for any $x > a$

$$\frac{H(x) - H(a)}{x-a} = \frac{H(x) - \Psi(a)}{x-a} \leq \frac{\Psi(x) - \Psi(a)}{x-a}$$

and

$$\frac{H(x) - H(a)}{x-a} \geq \frac{\phi(x) - \phi(a)}{x-a} .$$

It follows that

$$-\infty < \phi_2'(a) = \phi'(a) \leq DH(a+0) \leq \Psi'(a) = \Psi_2'(a) < +\infty$$

and similarly

$$-\infty < \Psi_1'(b) = \Psi'(b) \leq DH(b+0) \leq \phi'(b) = \phi_1'(b) < +\infty.$$

Thus by Theorem 17 the generalized solution H(x) is continuous on [a,b] and assumes the zero boundary values. $\square$

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